Multidimensional and Directional Filter Banks

Minh N. Do

Department of Electrical and Computer Engineering
University of Illinois at Urbana-Champaign

www.ifp.uiuc.edu/∼minhdo
minhdo@uiuc.edu

Joint work with Martin Vetterli (EPFL and UC Berkeley), Arthur Cunha, Yue Lu, Jianping Zhou (UIUC)

Outline

1. Multidimensional filter banks
   - Sampling lattices
   - Filter design
2. Laplacian pyramid frames
3. Iterated directional filter banks

1. Multidimensional Filter Banks

Motivation

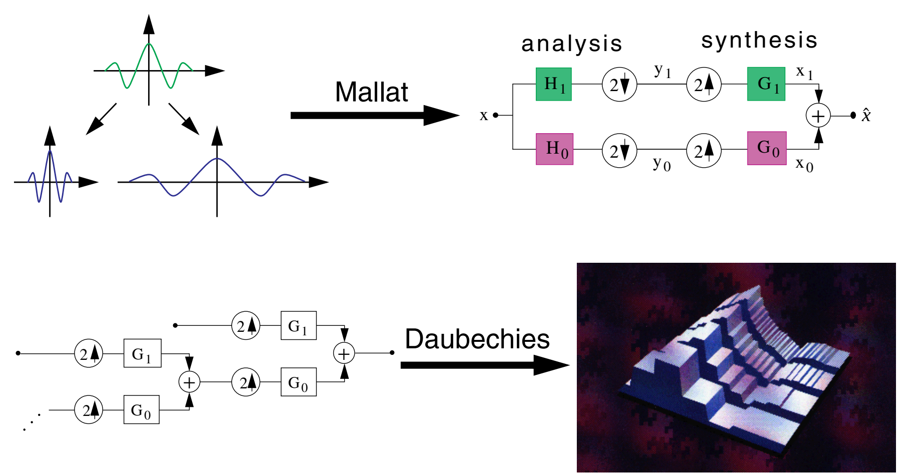
Looking for...

- Discrete-domain expansions for sampled data
- Structured transforms with fast algorithms
- Seamless connection with continuous-domain expansions

Answer: Filter banks and associated bases and frames.

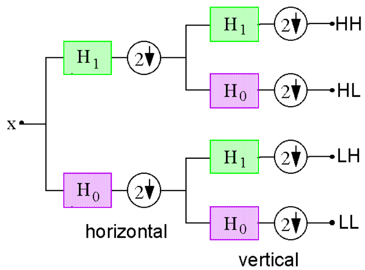
Filter Banks

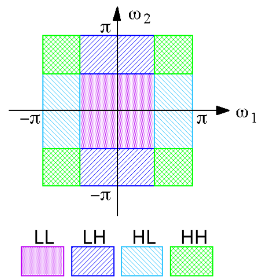
Wavelets and filter banks...
fundamental link between continuous and discrete domains



Multidimensional Filter Banks

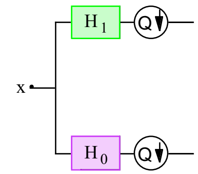
- **Common approach**: use 1-D techniques in a separable fashion

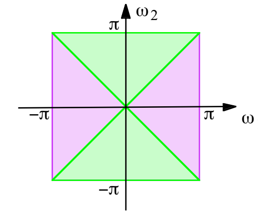




- **Limitations**:
  - very constrained filter design (separable filters)
  - only rectangular frequency partition possible

"True" Multidimensional Filter Banks

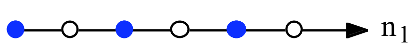




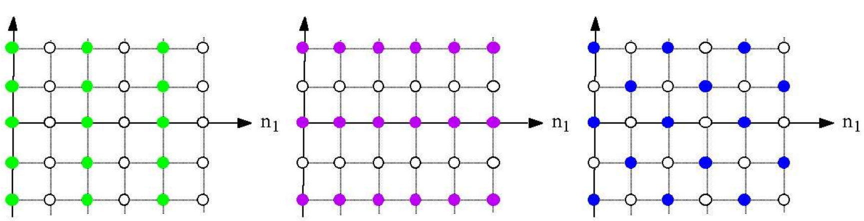
- **More flexibility**:
  - True multidimensional filters → directional filters
  - Sampling via lattices → discrete rotations

Sampling via Lattices

In 1D (downsample by 2):



In 2D (downsample by 2):



Multidimensional Signals and Filtering

- Discrete-time d-dimensional signal:

$$x[\boldsymbol{n}], \quad \boldsymbol{n} \stackrel{\text{def}}{=} (n_1, \ldots, n_d)^T \in \mathbb{Z}^d$$

- The z-transform:

$$X(\boldsymbol{z}) = \sum_{\boldsymbol{n} \in \mathbb{Z}^d} x[\boldsymbol{n}] \boldsymbol{z}^{-\boldsymbol{n}} \quad \text{where} \quad \boldsymbol{z}^{\boldsymbol{n}} = \prod_{i=1}^{d} z_i^{n_i}$$

- Filtering with $h[\boldsymbol{n}]$ yields

$$y[\boldsymbol{n}] = x[\boldsymbol{n}] * h[\boldsymbol{n}] = \sum_{\boldsymbol{k} \in \mathbb{Z}^d} x[\boldsymbol{k}] h[\boldsymbol{n} - \boldsymbol{k}]$$

$$Y(\boldsymbol{z}) = X(\boldsymbol{z}) H(\boldsymbol{z})$$

Multidimensional Sampling

Sampling operation is represented by a $d \times d$ nonsingular integer matrix $\boldsymbol{M}$.

Downsampling

$$x_d[\boldsymbol{n}] = x[\boldsymbol{M}\boldsymbol{n}].$$

Upsampling

$$x_u[\boldsymbol{n}] = \begin{cases} x[\boldsymbol{k}] & \text{if } \boldsymbol{n} = \boldsymbol{M}\boldsymbol{k},\ \boldsymbol{k} \in \mathbb{Z}^d \\ 0 & \text{otherwise.} \end{cases}$$

Sampling lattice

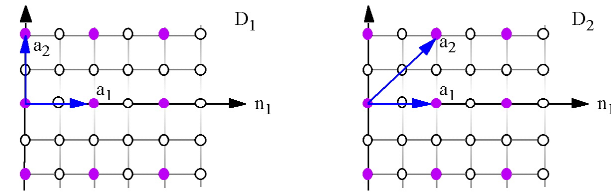
$$LAT(\boldsymbol{M}) \stackrel{\text{def}}{=} \{\boldsymbol{n} \ : \ \boldsymbol{n} = \boldsymbol{M}\boldsymbol{k},\ \boldsymbol{k} \in \mathbb{Z}^d\}$$
$$= \text{all linear combinations of columns of } \boldsymbol{M} \text{ with integer coefficients}$$

Sampling density is equal to $|\boldsymbol{M}| \stackrel{\text{def}}{=} |\det \boldsymbol{M}|$

Examples of Sampling Lattices

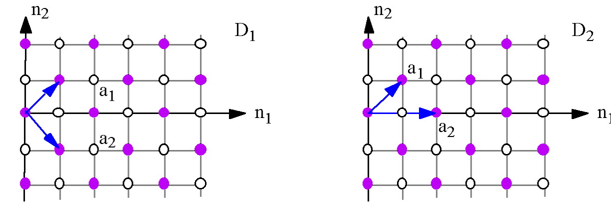
Separable lattice in two dimensions:

$$\boldsymbol{D}_1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad \boldsymbol{D}_2 = \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}$$



Quincunx lattice in two dimensions:

$$\boldsymbol{D}_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \boldsymbol{D}_2 = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$



Sampling Lattices and Matrices

Theorem 1. *$LAT(\boldsymbol{A}) = LAT(\boldsymbol{B})$ if and only if $\boldsymbol{A} = \boldsymbol{B}\boldsymbol{E}$ where $\boldsymbol{E}$ is a unimodular (i.e. $\det \boldsymbol{E} = \pm 1$) integer matrix.*

Example: Four basic unimodular matrices (shearing operators)

$$\boldsymbol{R}_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{R}_1 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{R}_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \boldsymbol{R}_3 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Theorem 2. [Smith form] *Any integer matrix $\boldsymbol{M}$ can be factorized as $\boldsymbol{M} = \boldsymbol{U}\boldsymbol{D}\boldsymbol{V}$ where $\boldsymbol{U}$ and $\boldsymbol{V}$ are unimodular integer matrices and $\boldsymbol{D}$ is an integer diagonal matrices.*

Example: For quincunx matrices

$$\boldsymbol{Q}_0 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \qquad \boldsymbol{Q}_1 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$\boldsymbol{Q}_0 = \boldsymbol{R}_1\boldsymbol{D}_0\boldsymbol{R}_2 = \boldsymbol{R}_2\boldsymbol{D}_1\boldsymbol{R}_1$ and $\boldsymbol{Q}_1 = \boldsymbol{R}_0\boldsymbol{D}_0\boldsymbol{R}_3 = \boldsymbol{R}_3\boldsymbol{D}_1\boldsymbol{R}_0$, where $\boldsymbol{D}_0 = \text{diag}(2, 1)$ and $\boldsymbol{D}_1 = \text{diag}(1, 2)$

Examples of Multidimensional Sampling

$\downarrow \boldsymbol{R}$ $\qquad$ $\downarrow \boldsymbol{Q}$

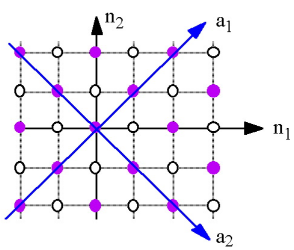
$$\boldsymbol{R}_3 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \qquad \boldsymbol{Q}_1 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \boldsymbol{R}_0\boldsymbol{D}_0\boldsymbol{R}_3$$

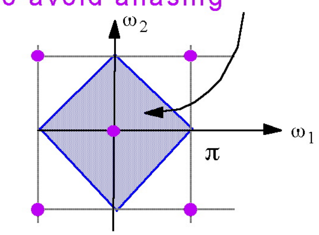
Multidimensional Sampling in Frequency-Domain (1/2)

Downsampling by M

$$X_d(\boldsymbol{\omega}) = \frac{1}{|\det(\boldsymbol{M})|} \sum_{\boldsymbol{k} \in \mathcal{N}(\boldsymbol{M}^T)} X(\boldsymbol{M}^{-T}(\boldsymbol{\omega} - 2\pi \boldsymbol{k})).$$

Here $\mathcal{N}(\boldsymbol{M})$ is the set of integer vectors of the form $\boldsymbol{M}\boldsymbol{t}$, $\boldsymbol{t} \in [0,1)^d$.





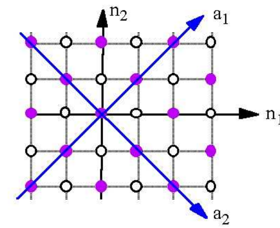
Multidimensional Sampling in Frequency-Domain (2/2)

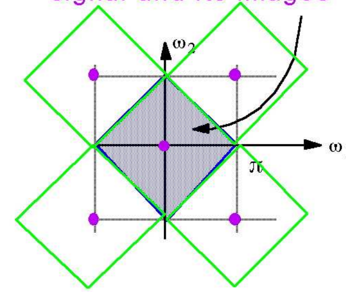
Upsampling by M

$$X_u(\boldsymbol{\omega}) = X(\boldsymbol{M}^T \boldsymbol{\omega}),$$
$$X_u(\boldsymbol{z}) = X(\boldsymbol{z}^{\boldsymbol{M}}).$$

Here $\boldsymbol{z}^{\boldsymbol{M}} \stackrel{\text{def}}{=} (\boldsymbol{z}^{\boldsymbol{m}_1}, \ldots, \boldsymbol{z}^{\boldsymbol{m}_d})^T$, where $\boldsymbol{m}_i$ is the i-th column of $\boldsymbol{M}$.

● samples from x[n]
○ inserted zeros





Polyphase Representation

- Polyphase components with respect to the sampling matrix $\boldsymbol{M}$:

$$x_i[\boldsymbol{n}] = x[\boldsymbol{M}\boldsymbol{n} + \boldsymbol{l}_i]$$

where $\{\boldsymbol{l}_i\}$ is the set of $|\boldsymbol{M}|$ integer vectors of the form $\boldsymbol{M}\boldsymbol{t}$, $\boldsymbol{t} \in [0,1)^d$

- Each polyphase component "lives" on a coset.

$$\mathcal{C}_i \stackrel{\text{def}}{=} \left\{ \boldsymbol{m} \;:\; \boldsymbol{m} = \boldsymbol{M}\boldsymbol{n} + \boldsymbol{l}_i,\; \boldsymbol{n} \in \mathbb{Z}^d \right\}, \quad 0 \le i \le |\boldsymbol{M}| - 1$$

$$\bigcup_{i=0}^{|\boldsymbol{M}|-1} \mathcal{C}_i = \mathbb{Z}^d, \qquad \mathcal{C}_i \bigcap \mathcal{C}_j = \emptyset,\; i \neq j.$$

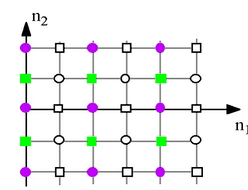
- In the z-domain:

$$X(\boldsymbol{z}) = \sum_{i=0}^{|\boldsymbol{M}|-1} \boldsymbol{z}^{-\boldsymbol{l}_i} X_i(\boldsymbol{z}^{\boldsymbol{M}})$$

Examples of Polyphase Representations

Separable lattice

$$\boldsymbol{M} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

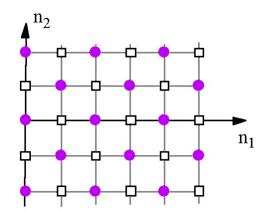


● polyphase component 0,0
□ polyphase component 1,0
■ polyphase component 0,1
○ polyphase component 1,1

$$X(z_1, z_2) = X_{00}(z_1^2, z_2^2) + z_1^{-1} X_{10}(z_1^2, z_2^2) + z_2^{-1} X_{01}(z_1^2, z_2^2) + z_1^{-1} z_2^{-1} X_{11}(z_1^2, z_2^2)$$

Quincunx lattice

$$\boldsymbol{M} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$



● polyphase component 0
□ polyphase component 1

$$X(z_1, z_2) = X_0(z_1 z_2, z_1 z_2^{-1}) + z_1^{-1} X_1(z_1 z_2, z_1 z_2^{-1})$$

Polyphase-Domain Analysis

Filter and downsample

$$x[\boldsymbol{n}] \longrightarrow \boxed{H(\boldsymbol{z})} \longrightarrow (\downarrow \boldsymbol{M}) \longrightarrow c[\boldsymbol{n}]$$

$$C(\boldsymbol{z}) = \sum_{i=0}^{|\boldsymbol{M}|-1} H_i(\boldsymbol{z}) X_i(\boldsymbol{z}) = \boldsymbol{H}(\boldsymbol{z}) x(\boldsymbol{z})$$

where $\boldsymbol{x}(z) \stackrel{\text{def}}{=} (X_0(z), \ldots, X_{|\boldsymbol{M}|-1}(z))^T$, and $\boldsymbol{H}(\boldsymbol{z}) \stackrel{\text{def}}{=} (H_0(\boldsymbol{z}), \ldots, H_{|\boldsymbol{M}|-1}(\boldsymbol{z}))$

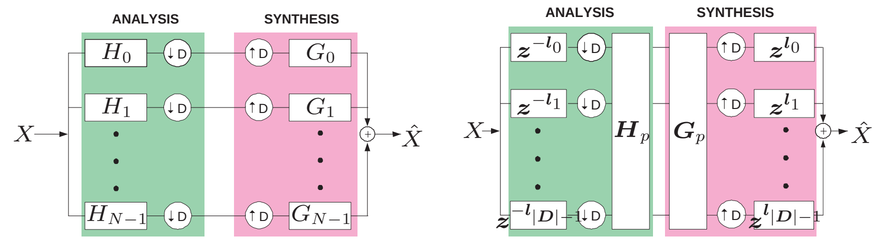
Upsample and filter

$$c[\boldsymbol{n}] \longrightarrow (\uparrow \boldsymbol{M}) \longrightarrow \boxed{G(\boldsymbol{z})} \longrightarrow p[\boldsymbol{n}]$$

$$P_i(\boldsymbol{z}) = G_i(\boldsymbol{z}) C(\boldsymbol{z}) \quad \text{or} \quad \boldsymbol{p}(\boldsymbol{z}) = \boldsymbol{G}(\boldsymbol{z}) C(\boldsymbol{z})$$

where $\boldsymbol{G}(\boldsymbol{z}) \stackrel{\text{def}}{=} (G_0(\boldsymbol{z}), \ldots, G_{|\boldsymbol{M}|-1}(\boldsymbol{z}))^T$, and $\boldsymbol{p}(\boldsymbol{z}) \stackrel{\text{def}}{=} (P_0(\boldsymbol{z}), \ldots, P_{|\boldsymbol{M}|-1}(\boldsymbol{z}))^T$

Filter Bank in the Polyphase-Domain



- **Critically sampled**: $|\boldsymbol{D}| = N$
- **Perfect reconstruction**: $\boldsymbol{G}_p(\boldsymbol{z}) \boldsymbol{H}_p(\boldsymbol{z}) = I_{|\boldsymbol{D}|}$
- **Biorthogonal**: Perfect reconstruction + critically sampled
- **Orthogonal**: $\boldsymbol{H}_p(\boldsymbol{z}) = \boldsymbol{G}_p^T(\boldsymbol{z}^{-1})$ and $\boldsymbol{G}_p(\boldsymbol{z}) \boldsymbol{G}_p^T(\boldsymbol{z}^{-1}) = I$ (i.e. $\boldsymbol{G}_p(\boldsymbol{z})$ is a paraunitary matrix)

A Reduction Theorem for Biorthogonal Filter Banks

Theorem 3. [ZhouD:04] *Suppose $\boldsymbol{G}(\boldsymbol{z})$ is an $N \times N$ matrix and $\boldsymbol{G}_{N-1}(\boldsymbol{z})$ is its submatrix obtained by deleting the last row of $\boldsymbol{G}(\boldsymbol{z})$. Suppose $\boldsymbol{H}(\boldsymbol{z})$ is another $N \times N$ matrix and $\boldsymbol{H}_{N-1}(\boldsymbol{z})$ is its submatrix obtained by by deleting the last column of $\boldsymbol{H}(\boldsymbol{z})$.*
Then $\boldsymbol{G}(\boldsymbol{z})\boldsymbol{H}(\boldsymbol{z}) = \boldsymbol{I}_N$ if and only if

$$\boldsymbol{G}_{N-1}(\boldsymbol{z}) \boldsymbol{H}_{N-1}(\boldsymbol{z}) = \boldsymbol{I}_{N-1}, \tag{1}$$

$$G_{N,i}(\boldsymbol{z}) = \alpha(\boldsymbol{z})(-1)^{i+N} \det \boldsymbol{H}_{i,N-1}(\boldsymbol{z}), \tag{2}$$

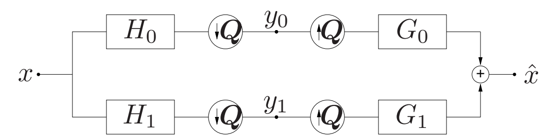
$$H_{i,N}(\boldsymbol{z}) = \alpha^{-1}(\boldsymbol{z})(-1)^{i+N} \det \boldsymbol{G}_{N-1,i}(\boldsymbol{z}), \tag{3}$$

where $\alpha(\boldsymbol{z}) = \det \boldsymbol{G}(\boldsymbol{z})$ is an arbitrary nonzero filter, $\boldsymbol{H}_{i,N-1}(\boldsymbol{z})$ is the submatrix of $\boldsymbol{H}_{N-1}(\boldsymbol{z})$ obtained by deleting its ith row, and $\boldsymbol{G}_{N-1,i}(\boldsymbol{z})$ is the submatrix of $\boldsymbol{G}_{N-1}(\boldsymbol{z})$ obtained by deleting its ith column.

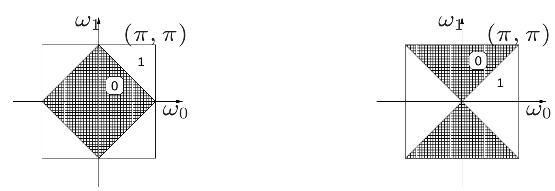
If $\boldsymbol{G}(\boldsymbol{z})$ and $\boldsymbol{H}(\boldsymbol{z})$ have FIR entries, then additionally $\alpha(\boldsymbol{z}) = \alpha \boldsymbol{z}^{\boldsymbol{k}}$

Corollary 1. *A N-channel biorthogonal FIR filter bank is completely determined by its first $N-1$ channels, and a scaled delay in the last channel.*

Example: Quincunx Filter Bank



Possible frequency partitions:



Equivalent biorthogonal condition for FIR filters:

$$H_0(\boldsymbol{z}) G_0(\boldsymbol{z}) + H_0(-\boldsymbol{z}) G_0(-\boldsymbol{z}) = 2, \quad \text{and}$$

$$H_1(\boldsymbol{z}) = \boldsymbol{z}^{-\boldsymbol{k}} G_0(-\boldsymbol{z}), \quad G_1(\boldsymbol{z}) = \boldsymbol{z}^{\boldsymbol{k}} H_0(-\boldsymbol{z}).$$

Filter Design Problem

Filter design amounts to solve

$$P(\boldsymbol{z}) + P(-\boldsymbol{z}) = 2, \qquad \text{(easy)}$$

where

$$P(\boldsymbol{z}) = H_0(\boldsymbol{z})G_0(\boldsymbol{z}) \qquad \text{(hard)}$$

Fundamental problem in multi-dimensions: no factorization theorem.

How to avoid MD factorization?

One possibility: Map 1D filters to MD filters...

Design via Mapping

1. Start with a 1D solution:

$$P^{(1D)}(z) + P^{(1D)}(-z) = 2 \quad \text{and} \quad P^{(1D)}(z) = H^{(1D)}(z)G^{(1D)}(z)$$

2. Apply a 1D to MD mapping to each filter:

$$F^{(1D)}(z) \rightarrow F^{(MD)}(\boldsymbol{z}) = F^{(1D)}(M(\boldsymbol{z}))$$

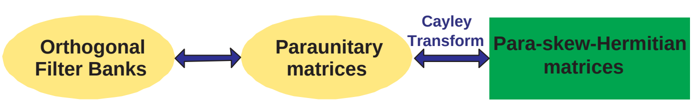
- If $M(\boldsymbol{z}) = -M(-\boldsymbol{z})$ then the perfect reconstruction condition is preserved

$$P^{(MD)}(\boldsymbol{z}) + P^{(MD)}(-\boldsymbol{z}) = P^{(1D)}(M(\boldsymbol{z})) + P^{(1D)}(M(-\boldsymbol{z})) = 2$$

- Can design the mapping to preserve and obtain other properties, like linear phase, vanishing moments, directional vanishing moments, ... (McClellan:73, TayK:93, ChenV:93, CunhaD:04)

This only works for biorthogonal filter banks...

MD Orthogonal FB Design using Cayley Transform



Definition. [Cayley transform] The Cayley transform of a matrix $\boldsymbol{U}(\boldsymbol{z})$ is

$$\boldsymbol{H}(\boldsymbol{z}) = \big(\boldsymbol{I} + \boldsymbol{U}(\boldsymbol{z})\big)^{-1}\big(\boldsymbol{I} - \boldsymbol{U}(\boldsymbol{z})\big).$$

Definition. $\boldsymbol{H}(\boldsymbol{z})$ is a para-skew-Hermitian (PSH) matrix if it satisfies

$$\boldsymbol{H}(\boldsymbol{z}^{-1}) = -\boldsymbol{H}^T(\boldsymbol{z}), \quad \text{for real coefficients.}$$

Proposition 1. [ZhouDK:04] *Suppose $\boldsymbol{U}(\boldsymbol{z})$ is an $N \times N$ matrix. Then there exists at least one $N \times N$ diagonal matrix Λ whose diagonal entries are either 1 or -1 such that $\boldsymbol{I} + \Lambda\boldsymbol{U}(\boldsymbol{z})$ is nonsingular.*

Theorem 4. [ZhouDK:04] *The Cayley transform of a paraunitary matrix is a PSH matrix. Conversely, the Cayley transform of a PSH matrix is a paraunitary matrix.*

Example: Two-Channel Filter Banks

Polyphase-domain:

$$\boldsymbol{U}(\boldsymbol{z}) = \begin{pmatrix} U_{00}(\boldsymbol{z}) & U_{01}(\boldsymbol{z}) \\ U_{10}(\boldsymbol{z}) & U_{11}(\boldsymbol{z}) \end{pmatrix}$$

The paraunitary condition
$\boldsymbol{U}(\boldsymbol{z})\boldsymbol{U}^T(\boldsymbol{z}^{-1}) = \boldsymbol{I}$
becomes

$$\begin{aligned} U_{00}(\boldsymbol{z})U_{00}(\boldsymbol{z}^{-1}) + U_{01}(\boldsymbol{z})U_{01}(\boldsymbol{z}^{-1}) &= 1, \\ U_{00}(\boldsymbol{z})U_{10}(\boldsymbol{z}^{-1}) + U_{01}(\boldsymbol{z})U_{11}(\boldsymbol{z}^{-1}) &= 0, \\ U_{10}(\boldsymbol{z})U_{10}(\boldsymbol{z}^{-1}) + U_{11}(\boldsymbol{z})U_{11}(\boldsymbol{z}^{-1}) &= 1. \end{aligned}$$

$\rightarrow$ nonlinear equations

Cayley-domain:

$$\boldsymbol{H}(\boldsymbol{z}) = \begin{pmatrix} H_{00}(\boldsymbol{z}) & H_{01}(\boldsymbol{z}) \\ H_{10}(\boldsymbol{z}) & H_{11}(\boldsymbol{z}) \end{pmatrix}.$$

The PSH condition
$\boldsymbol{H}(\boldsymbol{z}^{-1}) = -\boldsymbol{H}^T(\boldsymbol{z})$
becomes

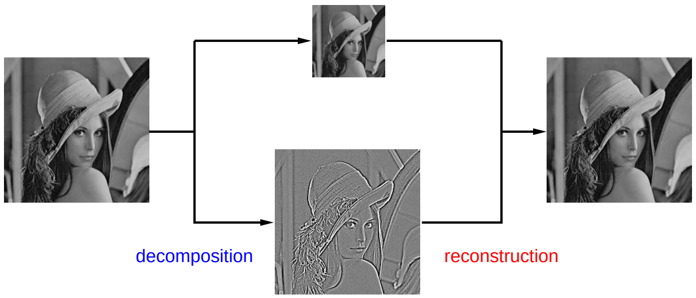
$$\begin{aligned} H_{00}(\boldsymbol{z}^{-1}) &= -H_{00}(\boldsymbol{z}), \\ H_{11}(\boldsymbol{z}^{-1}) &= -H_{11}(\boldsymbol{z}), \\ H_{01}(\boldsymbol{z}^{-1}) &= -H_{10}(\boldsymbol{z}). \end{aligned}$$

$\rightarrow$ linear equations

Outline

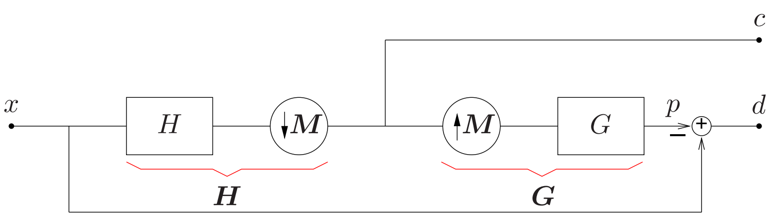
1. Multidimensional filter banks
2. Laplacian pyramid frames
   - Framing pyramids
   - New reconstruction (dual frame)
3. Iterated directional filter banks

2. Laplacian Pyramid Frames



- **Reason:** avoid "frequency scrambling" due to $(\downarrow)$ of the HP channel.
- Laplacian pyramid as a frame operator → tight frame exists.
- New reconstruction: efficient filter bank for *dual* frame (pseudo-inverse).

Decomposition in the Laplacian Pyramid

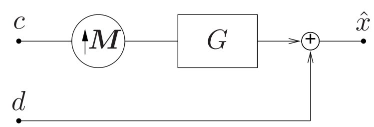


$$C(z) = \boldsymbol{H}(z)\boldsymbol{x}(z)$$
$$\boldsymbol{d}(z) = \boldsymbol{x}(z) - \boldsymbol{G}(z)\boldsymbol{H}(z)\boldsymbol{x}(z) = (\boldsymbol{I} - \boldsymbol{G}(z)\boldsymbol{H}(z))\boldsymbol{x}(z).$$

Combining gives

$$\underbrace{\begin{pmatrix} C(z) \\ \boldsymbol{d}(z) \end{pmatrix}}_{\boldsymbol{y}(z)} = \underbrace{\begin{pmatrix} \boldsymbol{H}(z) \\ \boldsymbol{I} - \boldsymbol{G}(z)\boldsymbol{H}(z) \end{pmatrix}}_{\boldsymbol{A}(z)} \boldsymbol{x}(z).$$

Usual Reconstruction in the Laplacian Pyramid



$$\hat{\boldsymbol{x}}(z) = \boldsymbol{G}(z)C(z) + \boldsymbol{d}(z)$$

Or,

$$\hat{\boldsymbol{x}}(z) = \underbrace{\begin{pmatrix} \boldsymbol{G}(z) & \boldsymbol{I} \end{pmatrix}}_{\boldsymbol{S}_1(z)} \underbrace{\begin{pmatrix} C(z) \\ \boldsymbol{d}(z) \end{pmatrix}}_{\boldsymbol{y}(z)}.$$

Note that $\boldsymbol{S}_1(z)\boldsymbol{A}(z) = \boldsymbol{I}$ (perfect reconstruction) for *any* $\boldsymbol{H}(z)$ and $\boldsymbol{G}(z)$.

But... what about noisy pyramids: $\hat{\boldsymbol{y}} = \boldsymbol{y} + \boldsymbol{e}$?

Frame Analysis

- LP is a frame operator ($\boldsymbol{A}$) with redundancy.
- It admits an infinite number of left inverses.
- Let $\boldsymbol{S}$ be an arbitrary left inverse of $\boldsymbol{A}$,

$$\hat{\boldsymbol{x}} = \boldsymbol{S}\hat{\boldsymbol{y}} = \boldsymbol{S}(\boldsymbol{y} + \boldsymbol{e}) = \boldsymbol{x} + \boldsymbol{S}\boldsymbol{e}.$$

- The optimal left inverse (minimizing $\|\boldsymbol{S}\|$) is the pseudo-inverse (or dual frame) of $\boldsymbol{A}$:

$$\boldsymbol{A}^{\dagger} = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T.$$

- If the noise is white, then among all left inverses, the pseudo-inverse minimizes the reconstruction MSE.

A Tight Frame Pyramid

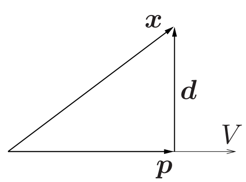
Proposition. *The Laplacian pyramid with orthogonal filters is a tight frame with frame bounds equal to 1.*

Proof: Orthogonal condition means $\boldsymbol{G}^*(z)\boldsymbol{G}(z) = 1$ and $\boldsymbol{H}(z) = \boldsymbol{G}^*(z)$. With this, we can directly verify that

$$\boldsymbol{A}^*(z)\boldsymbol{A}(z) = \begin{pmatrix} \boldsymbol{H}^*(z) & \boldsymbol{I} - \boldsymbol{H}^*(z)\boldsymbol{G}^*(z) \end{pmatrix} \begin{pmatrix} \boldsymbol{H}(z) \\ \boldsymbol{I} - \boldsymbol{G}(z)\boldsymbol{H}(z) \end{pmatrix} = \boldsymbol{I}.$$

Geometrical interpretation:

$$p[n] = \sum_{k \in \mathbb{Z}^d} \underbrace{\langle x[\bullet], g[\bullet - \boldsymbol{M}k] \rangle}_{c[k]} g[n - \boldsymbol{M}k].$$



Using the *Pythagorean* theorem:

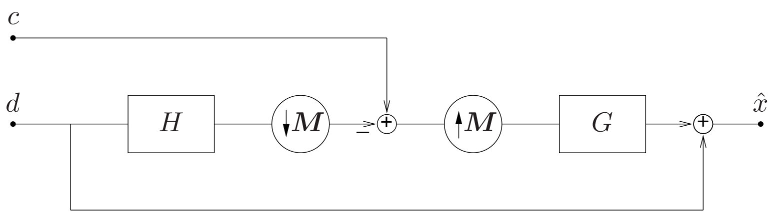
$$\|\boldsymbol{x}\|^2 = \|\boldsymbol{p}\|^2 + \|\boldsymbol{d}\|^2 = \|\boldsymbol{c}\|^2 + \|\boldsymbol{d}\|^2.$$

New Reconstruction for the Laplacian Pyramid

For tight frame with frame bounds 1, the pseudo-inverse of $\boldsymbol{A}(\boldsymbol{z})$ is

$$\boldsymbol{A}^{\dagger}(\boldsymbol{z}) = \boldsymbol{A}^T(\boldsymbol{z}) = \begin{pmatrix} \boldsymbol{H}(\boldsymbol{z}) \\ \boldsymbol{I} - \boldsymbol{G}(\boldsymbol{z})\boldsymbol{G}^T(\boldsymbol{z}) \end{pmatrix}^T = \begin{pmatrix} \boldsymbol{G}(\boldsymbol{z}) & \boldsymbol{I} - \boldsymbol{G}(\boldsymbol{z})\boldsymbol{G}^T(\boldsymbol{z}) \end{pmatrix}.$$

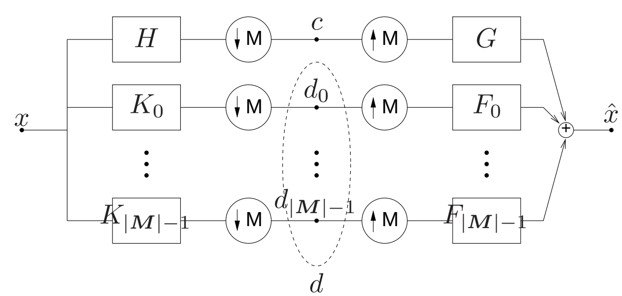
Which leads to the optimal reconstruction:



This also works for biorthogonal filters H and G.

Laplacian Pyramids as an Oversampled Filter Bank

From the polyphase representation of $\boldsymbol{A}(\boldsymbol{z})$ and $\boldsymbol{S}(\boldsymbol{z})$



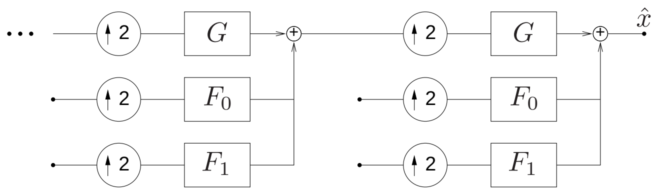
For the usual reconstruction (REC-1)

$$F_i^{[1]}(\boldsymbol{z}) = \boldsymbol{z}^{-\boldsymbol{k}_i}, \quad \text{for } i = 0, \ldots, |\boldsymbol{M}| - 1.$$

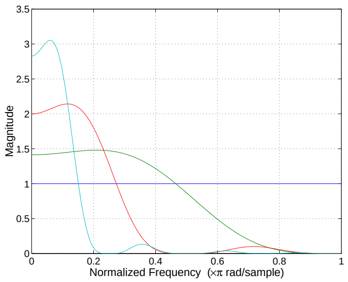
For the new reconstruction (REC-2)

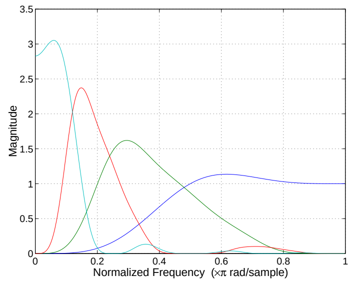
$$F_i^{[2]}(\boldsymbol{z}) = \boldsymbol{z}^{-\boldsymbol{k}_i} - G(\boldsymbol{z})H_i(\boldsymbol{z}^{\boldsymbol{M}}), \quad \text{for } i = 0, \ldots, |\boldsymbol{M}| - 1.$$

Multilevel Laplacian Pyramids



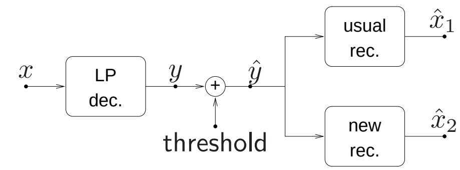
Frequency responses of equivalent synthesis filters: REC-1 and REC-2





The new reconstruction is crucial for building multiscale directional frames.

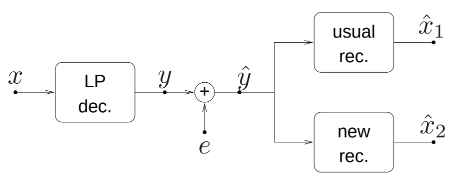
Experimental Results (1/2)



Nonlinear approximation: SNR's of the reconstructed images from the M most significant LP coefficients (after 6 levels of decomposition)

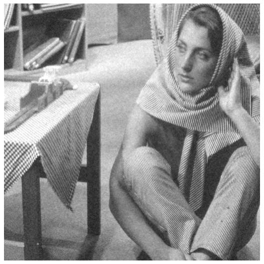
M		2^{12}	2^{14}	2^{16}
Barbara	REC-1	9.68	12.56	20.94
	REC-2	**9.87**	**13.18**	**21.75**
Goldhill	REC-1	12.30	15.79	21.55
	REC-2	**12.60**	**16.23**	**22.19**
Peppers	REC-1	15.06	20.81	26.77
	REC-2	**15.62**	**21.33**	**27.32**

Experimental Results (2/2)



With additive uniform white noise in $[0,\ 0.1]$ (non-zero mean)

usual rec.
SNR = 6.28 dB



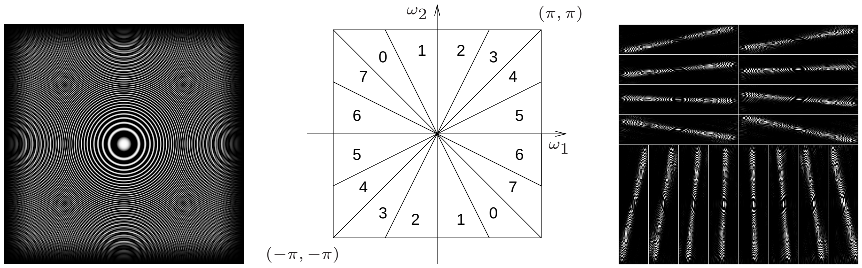
new rec.
SNR = 17.42 dB

Outline

1. Multidimensional filter banks
2. Laplacian pyramid frames
3. Iterated directional filter banks
   - How to obtain directional decomposition with 2D tree-structured filter banks
   - Local directional bases

3. Iterated Directional Filter Banks

- **Feature:** division of 2-D spectrum into fine slices using tree-structured filter banks.

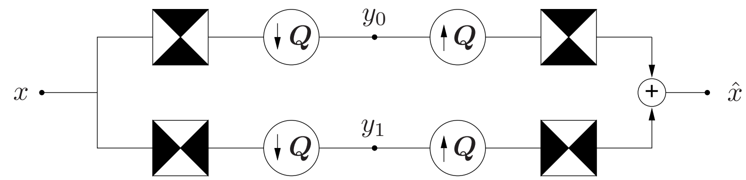


- **Background:** Bamberger and Smith ('92) cleverly used quincunx FB's, modulation and shearing.

- **We propose:**
  - a simplified DFB with fan FB's and shearing
  - use DFB to construct directional bases

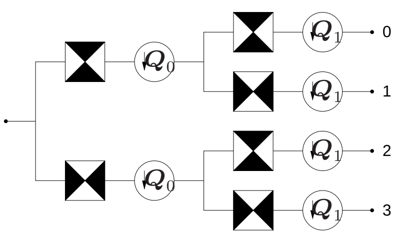
Our Simplified DFB: Two Building Blocks

- Frequency splitting by the quincunx filter banks (Vetterli'84).



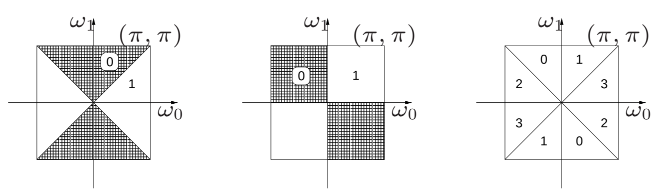
- Shearing by resampling

Illustration: The First Two Levels of the DFB

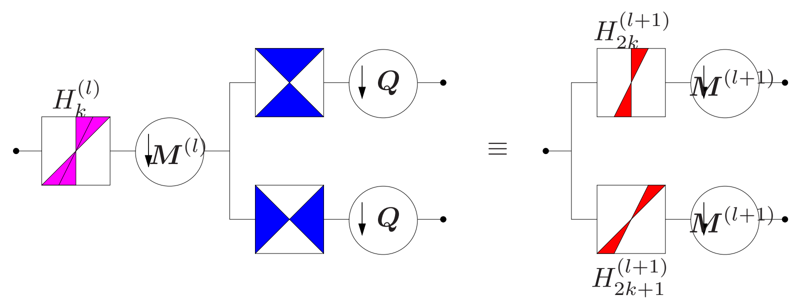


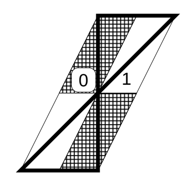
Using the multirate identity:

The support configurations of equivalent filters are:



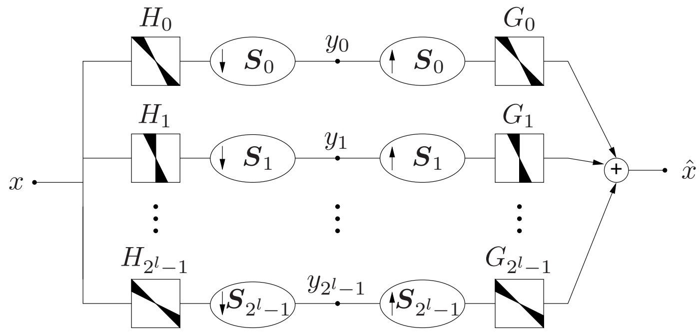
How Frequency is Divided into Finer Direction?





Overall sampling: $M_k^{(l)} = [2 \cdot D_i^{l-2}] \cdot R^{s_l(k)}$
$\equiv$ separable sampling, then shearing (*doesn't change lattic*

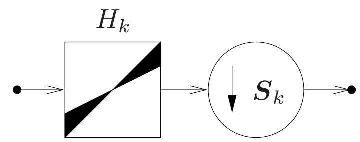
Multichannel View of the Directional Filter Bank

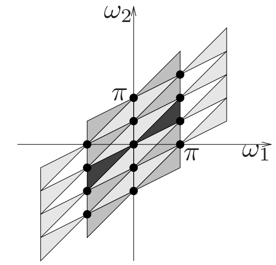


Use two *separable* sampling matrices:

$$S_k = \begin{cases} \begin{bmatrix} 2^{l-1} & 0 \\ 0 & 2 \end{bmatrix} & 0 \le k < 2^{l-1} \quad (\text{"near horizontal" direction}) \\ \begin{bmatrix} 2 & 0 \\ 0 & 2^{l-1} \end{bmatrix} & 2^{l-1} \le k < 2^l \quad (\text{"near vertical" direction}) \end{cases}$$

Why Critical Sampling Works?





Frequency tiling in the MDFB:

$$\sum_{m\in\mathbb{Z}^2} \delta_{R_k}(\omega - 2\pi S_k^{-T} m) = 1 \quad \text{for all} \quad k = 0,\dots,2^l-1;\ \omega \in \mathbb{R}^2.$$

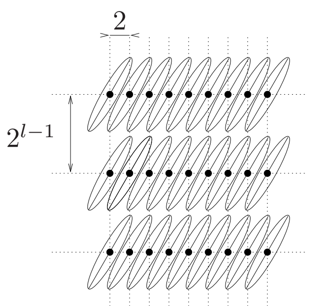
where R_k is the ideal frequency region for H_k.

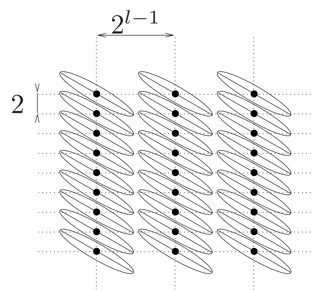
General Bases from the DFB

An l-levels DFB creates a local directional basis of $l^2(\mathbb{Z}^2)$:

$$\left\{ g_k^{(l)}[\cdot - S_k^{(l)} n] \right\}_{0\le k<2^l,\ n\in\mathbb{Z}^2}$$

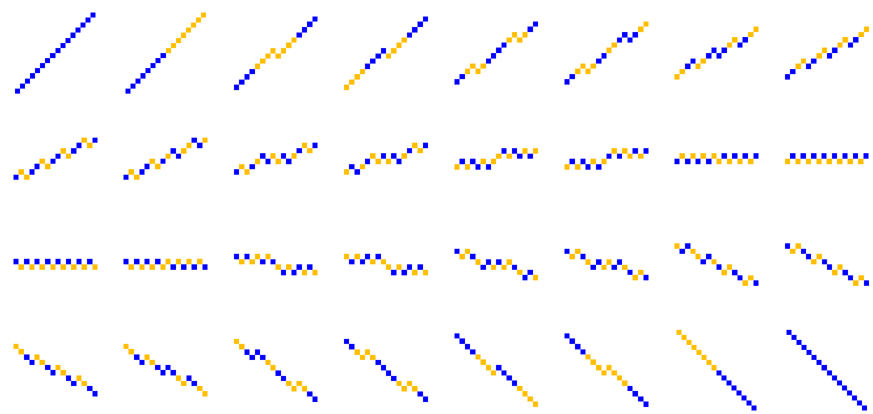
- $G_k^{(l)}$ are directional filters:
- Sampling lattices (spatial tiling):





Example of DFB Impulse Responses

32 equivalent filters for the first half channels (basically horizontal directions) of a 6-levels DFB that use the Haar filters



References

- Multidimensional filter banks
  - P. P. Vaidyanathan, *Multirate Systems and Filter Banks*, Prentice-Hall, 1993.
  - M. Vetterli and J. Kovačević, *Wavelets and Subband Coding*, Prentice-Hall, 1995.
  - J. Zhou, M. N. Do, and J. Kovačević, Multidimensional orthogonal filter bank characterization and design using the Cayley transform, *IEEE Trans. on Image Proc.*, to appear.

- Laplacian pyramid frames
  - M. N. Do and M. Vetterli, "Framing pyramids," *IEEE Trans. on Signal Proc.*, pp. 2329-2342, Sep. 2003.

- Iterated directional filter banks
  - M. N. Do, "Directional multiresolution image representations," Ph.D. thesis, EPFL, December 2001. (Chapter 3)

- Software and downloadable papers: www.ifp.uiuc.edu/∼minhdo