

Multidimensional and Directional Filter Banks

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Outline

1. Multidimensional filter banks
 - Sampling lattices
 - Filter design
2. Laplacian pyramid frames
3. Iterated directional filter banks

1. Multidimensional Filter Banks

Motivation

Looking for...

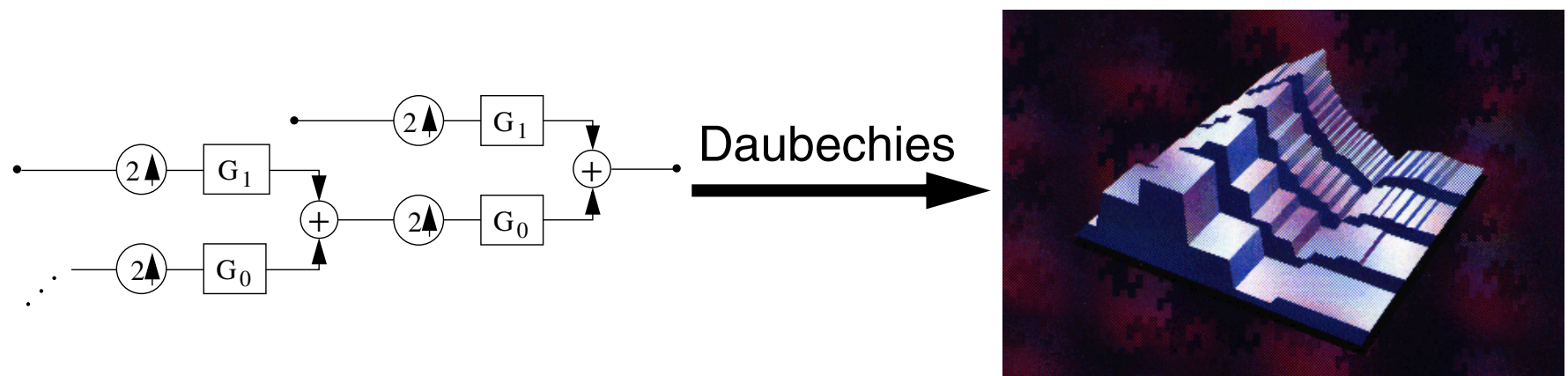
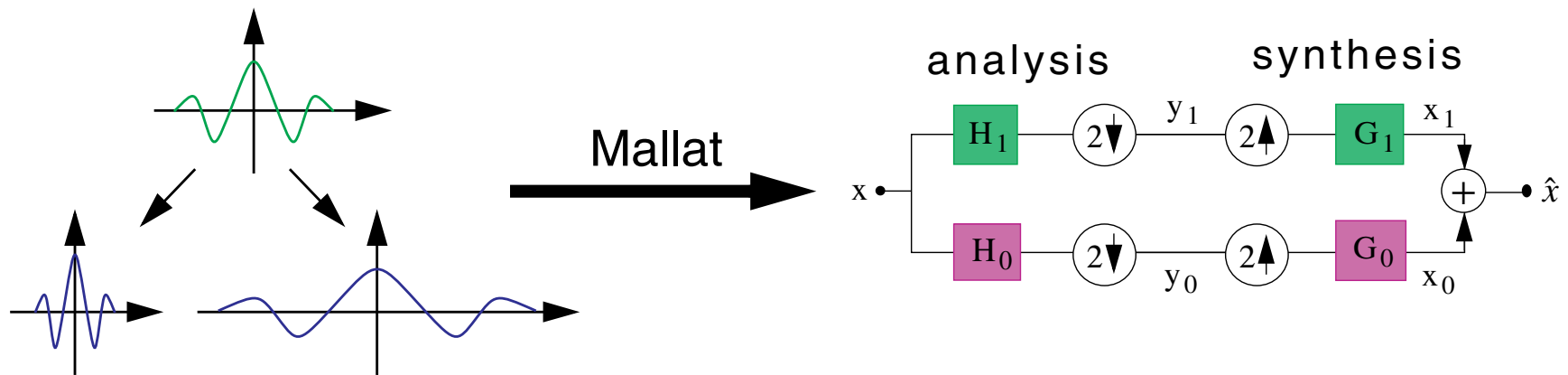
- Discrete-domain expansions for **sampled data**
- Structured transforms with **fast algorithms**
- **Seamless connection** with continuous-domain expansions

Answer: **Filter banks and associated bases and frames.**

Filter Banks

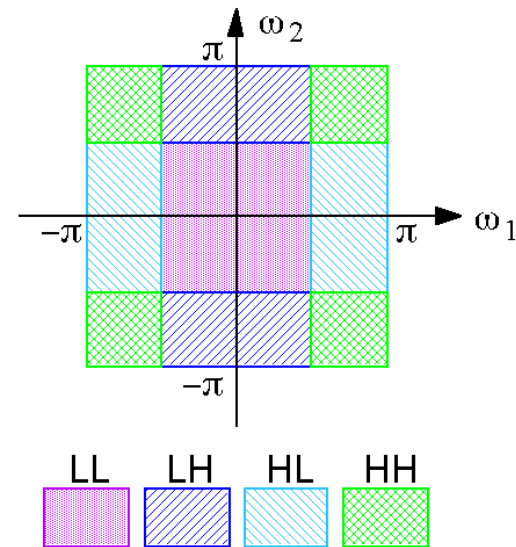
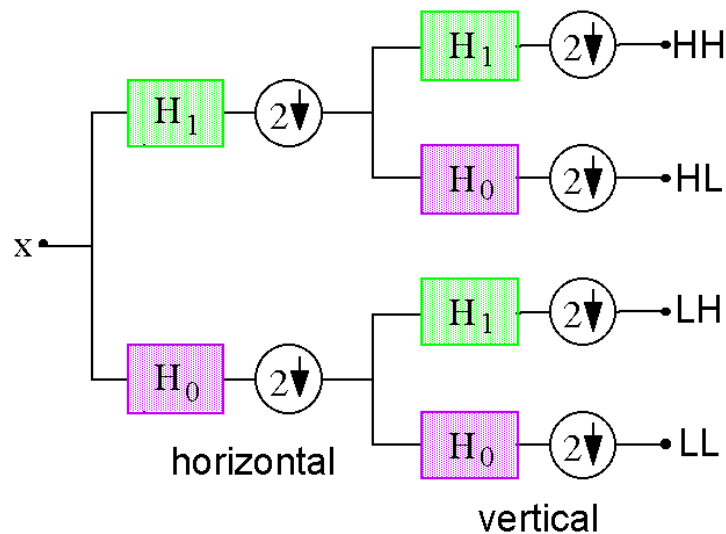
Wavelets and filter banks...

fundamental link between **continuous** and **discrete** domains



Multidimensional Filter Banks

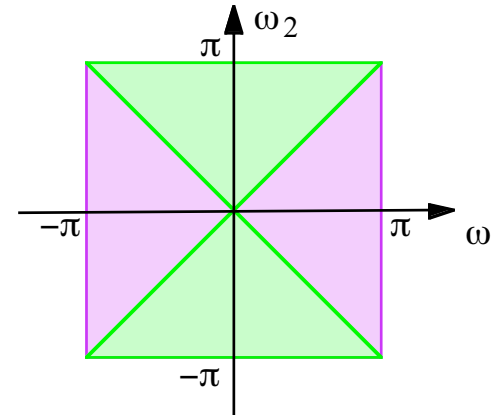
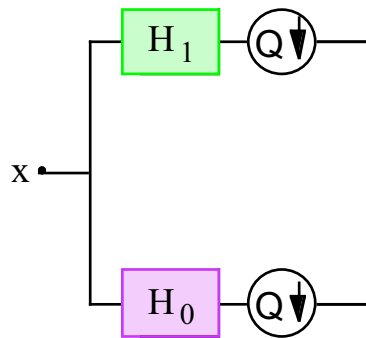
- **Common approach:** use 1-D techniques in a **separable** fashion



- **Limitations:**

- very constrained filter design (separable filters)
- only rectangular frequency partition possible

“True” Multidimensional Filter Banks



- **More flexibility:**

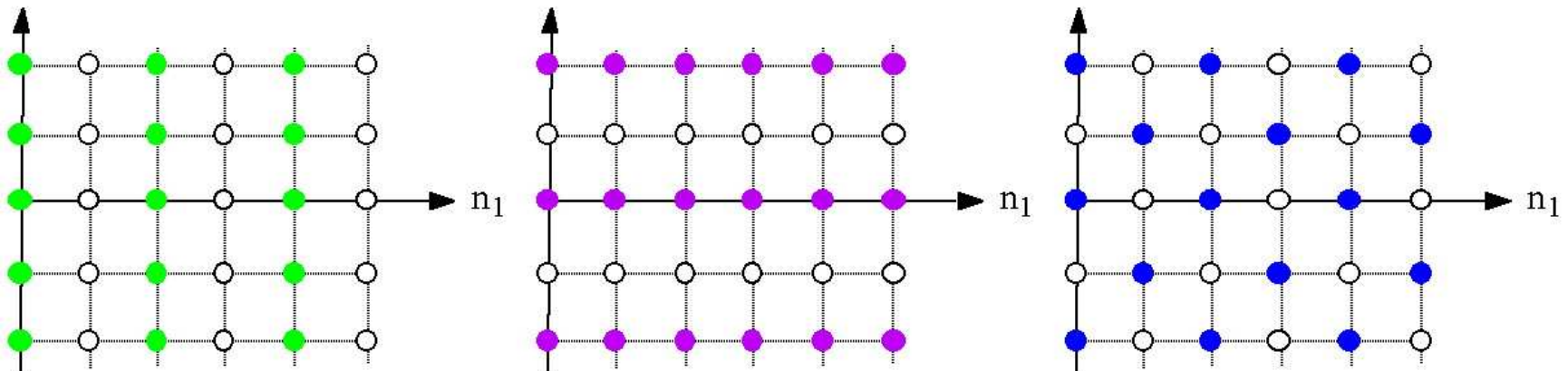
- True multidimensional filters \rightarrow directional filters
- Sampling via lattices \rightarrow discrete rotations

Sampling via Lattices

In 1D (downsample by 2):



In 2D (downsample by 2):



Multidimensional Signals and Filtering

- Discrete-time d -dimensional signal:

$$x[\mathbf{n}], \quad \mathbf{n} \stackrel{\text{def}}{=} (n_1, \dots, n_d)^T \in \mathbb{Z}^d$$

- The z -transform:

$$X(\mathbf{z}) = \sum_{\mathbf{n} \in \mathbb{Z}^d} x[\mathbf{n}] z^{-\mathbf{n}} \quad \text{where} \quad z^{\mathbf{n}} = \prod_{i=1}^d z_i^{n_i}$$

- Filtering with $h[\mathbf{n}]$ yields

$$y[\mathbf{n}] = x[\mathbf{n}] * h[\mathbf{n}] = \sum_{\mathbf{k} \in \mathbb{Z}^d} x[\mathbf{k}] h[\mathbf{n} - \mathbf{k}]$$

$$Y(\mathbf{z}) = X(\mathbf{z})H(\mathbf{z})$$

Multidimensional Sampling

Sampling operation is represented by a $d \times d$ nonsingular integer matrix M .

Downsampling

$$x_d[\mathbf{n}] = x[M\mathbf{n}].$$

Upsampling

$$x_u[\mathbf{n}] = \begin{cases} x[\mathbf{k}] & \text{if } \mathbf{n} = M\mathbf{k}, \mathbf{k} \in \mathbb{Z}^d \\ 0 & \text{otherwise.} \end{cases}$$

Sampling lattice

$$LAT(M) \stackrel{\text{def}}{=} \{ \mathbf{n} : \mathbf{n} = M\mathbf{k}, \mathbf{k} \in \mathbb{Z}^d \}$$

= all linear combinations of columns of M with integer coefficients

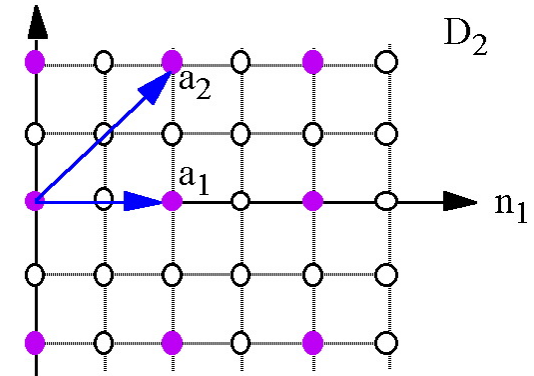
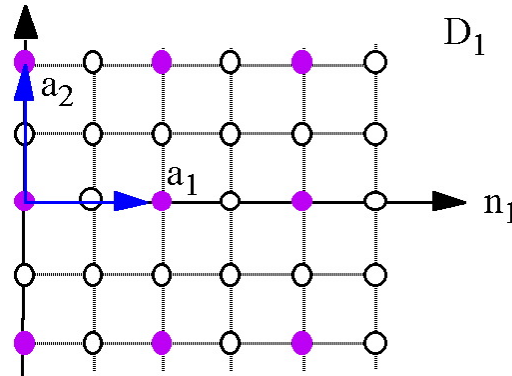
Sampling density is equal to $|M| \stackrel{\text{def}}{=} |\det M|$

Examples of Sampling Lattices

Separable lattice in two dimensions:

$$D_1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

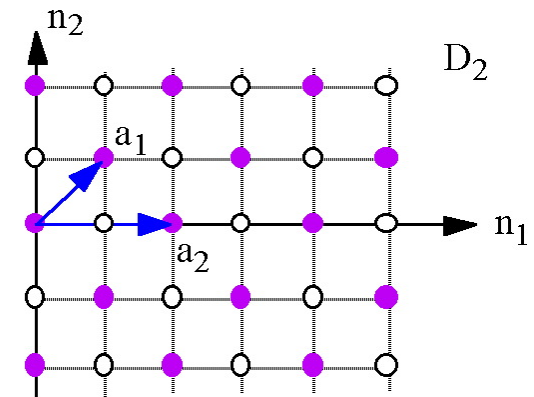
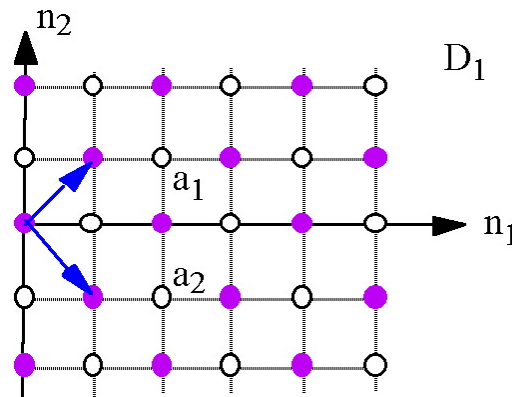
$$D_2 = \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}$$



Quincunx lattice in two dimensions:

$$D_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$D_2 = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$



Sampling Lattices and Matrices

Theorem 1. $LAT(\mathbf{A}) = LAT(\mathbf{B})$ if and only if $\mathbf{A} = \mathbf{B}\mathbf{E}$ where \mathbf{E} is a unimodular (i.e. $\det \mathbf{E} = \pm 1$) integer matrix.

Example: Four basic unimodular matrices (**shearing** operators)

$$\mathbf{R}_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{R}_1 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{R}_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{R}_3 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Theorem 2. [Smith form] Any integer matrix \mathbf{M} can be factorized as $\mathbf{M} = \mathbf{U}\mathbf{D}\mathbf{V}$ where \mathbf{U} and \mathbf{V} are unimodular integer matrices and \mathbf{D} is an integer diagonal matrices.

Example: For quincunx matrices

$$\mathbf{Q}_0 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{Q}_1 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$\mathbf{Q}_0 = \mathbf{R}_1\mathbf{D}_0\mathbf{R}_2 = \mathbf{R}_2\mathbf{D}_1\mathbf{R}_1$ and $\mathbf{Q}_1 = \mathbf{R}_0\mathbf{D}_0\mathbf{R}_3 = \mathbf{R}_3\mathbf{D}_1\mathbf{R}_0$,
where $\mathbf{D}_0 = \text{diag}(2, 1)$ and $\mathbf{D}_1 = \text{diag}(1, 2)$

Examples of Multidimensional Sampling



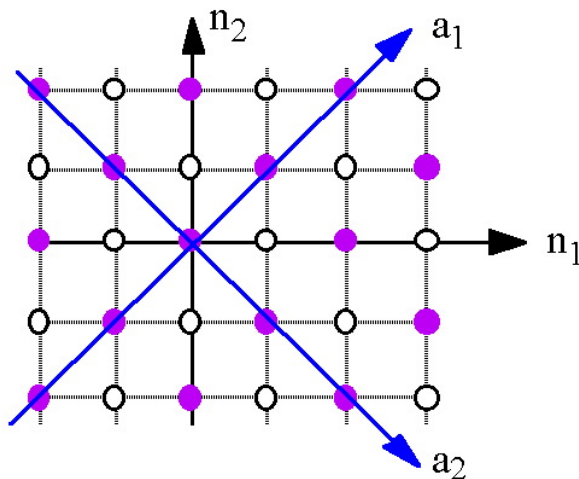
$$R_3 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \quad Q_1 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = R_0 D_0 R_3$$

Multidimensional Sampling in Frequency-Domain (1/2)

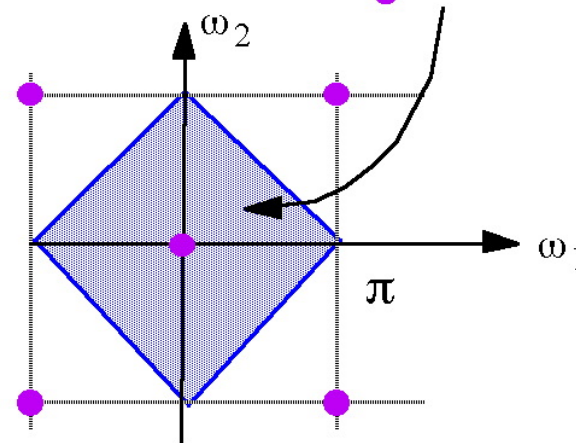
Downsampling by M

$$X_d(\boldsymbol{\omega}) = \frac{1}{|\det(\mathbf{M})|} \sum_{\mathbf{k} \in \mathcal{N}(\mathbf{M}^T)} X(\mathbf{M}^{-T}(\boldsymbol{\omega} - 2\pi\mathbf{k})).$$

Here $\mathcal{N}(\mathbf{M})$ is the set of integer vectors of the form $\mathbf{M}\mathbf{t}$, $\mathbf{t} \in [0, 1)^d$.



to avoid aliasing



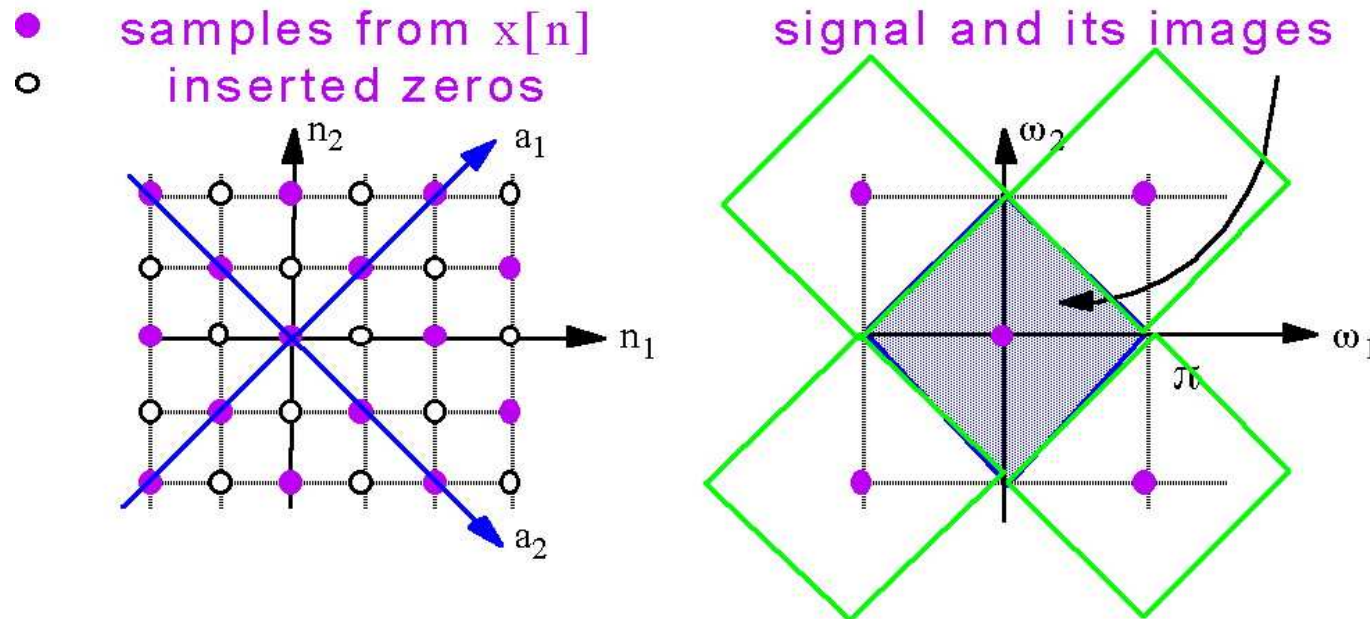
Multidimensional Sampling in Frequency-Domain (2/2)

Upsampling by M

$$X_u(\boldsymbol{\omega}) = X(\mathbf{M}^T \boldsymbol{\omega}),$$

$$X_u(\mathbf{z}) = X(\mathbf{z}^{\mathbf{M}}).$$

Here $\mathbf{z}^{\mathbf{M}} \stackrel{\text{def}}{=} (z^{m_1}, \dots, z^{m_d})^T$, where m_i is the i -th column of \mathbf{M} .



Polyphase Representation

- Polyphase components with respect to the sampling matrix \mathbf{M} :

$$x_i[\mathbf{n}] = x[\mathbf{M}\mathbf{n} + \mathbf{l}_i]$$

where $\{\mathbf{l}_i\}$ is the set of $|\mathbf{M}|$ integer vectors of the form $\mathbf{M}\mathbf{t}$, $\mathbf{t} \in [0, 1)^d$

- Each polyphase component “lives” on a **coset**.

$$\mathcal{C}_i \stackrel{\text{def}}{=} \{ \mathbf{m} : \mathbf{m} = \mathbf{M}\mathbf{n} + \mathbf{l}_i, \mathbf{n} \in \mathbb{Z}^d \}, \quad 0 \leq i \leq |\mathbf{M}| - 1$$

$$\bigcup_{i=0}^{|\mathbf{M}|-1} \mathcal{C}_i = \mathbb{Z}^d, \quad \mathcal{C}_i \cap \mathcal{C}_j = \emptyset, \quad i \neq j.$$

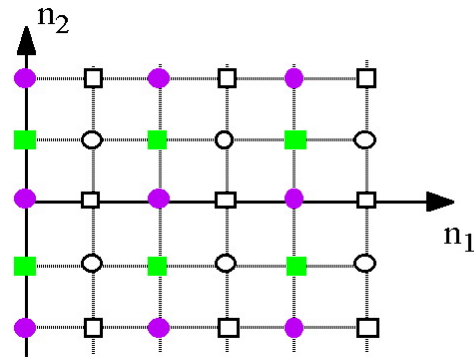
- In the z -domain:

$$X(\mathbf{z}) = \sum_{i=0}^{|\mathbf{M}|-1} \mathbf{z}^{-\mathbf{l}_i} X_i(\mathbf{z}^{\mathbf{M}})$$

Examples of Polyphase Representations

Separable lattice

$$M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

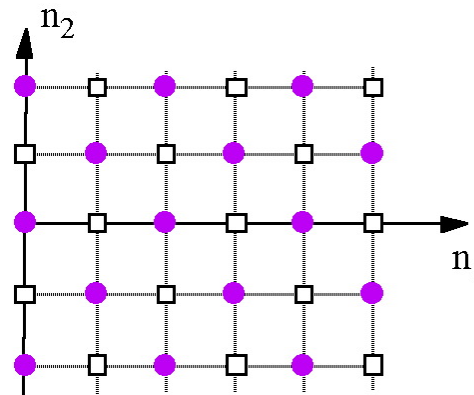


- polyphase component 0,0
- polyphase component 1,0
- polyphase component 0,1
- polyphase component 1,1

$$X(z_1, z_2) = X_{00}(z_1^2, z_2^2) + z_1^{-1} X_{10}(z_1^2, z_2^2) + z_2^{-1} X_{01}(z_1^2, z_2^2) + z_1^{-1} z_2^{-1} X_{11}(z_1^2, z_2^2)$$

Quincunx lattice

$$M = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$



- polyphase component 0
- polyphase component 1

$$X(z_1, z_2) = X_0(z_1 z_2, z_1 z_2^{-1}) + z_1^{-1} X_1(z_1 z_2, z_1 z_2^{-1})$$

Polyphase-Domain Analysis

Filter and downsample

$$x[\mathbf{n}] \longrightarrow \boxed{H(\mathbf{z})} \longrightarrow (\downarrow M) \longrightarrow c[\mathbf{n}]$$

$$C(\mathbf{z}) = \sum_{i=0}^{|\mathbf{M}|-1} H_i(\mathbf{z})X_i(\mathbf{z}) = \mathbf{H}(\mathbf{z})\mathbf{x}(\mathbf{z})$$

where $\mathbf{x}(\mathbf{z}) \stackrel{\text{def}}{=} (X_0(\mathbf{z}), \dots, X_{|\mathbf{M}|-1}(\mathbf{z}))^T$, and $\mathbf{H}(\mathbf{z}) \stackrel{\text{def}}{=} (H_0(\mathbf{z}), \dots, H_{|\mathbf{M}|-1}(\mathbf{z}))$

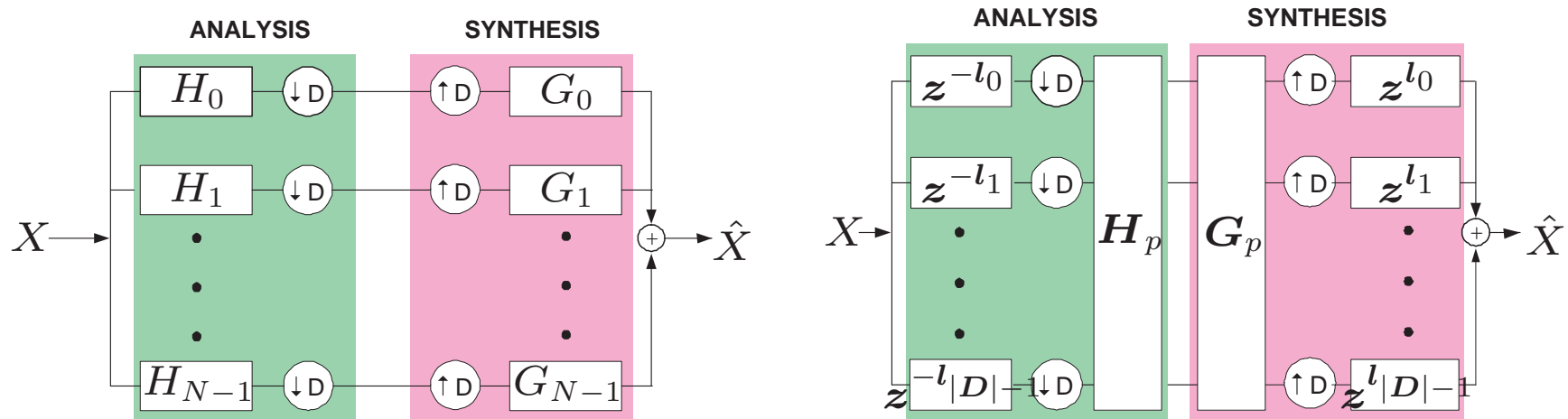
Upsample and filter

$$c[\mathbf{n}] \longrightarrow (\uparrow M) \longrightarrow \boxed{G(\mathbf{z})} \longrightarrow p[\mathbf{n}]$$

$$P_i(\mathbf{z}) = G_i(\mathbf{z})C(\mathbf{z}) \quad \text{or} \quad \mathbf{p}(\mathbf{z}) = \mathbf{G}(\mathbf{z})C(\mathbf{z})$$

where $\mathbf{G}(\mathbf{z}) \stackrel{\text{def}}{=} (G_0(\mathbf{z}), \dots, G_{|\mathbf{M}|-1}(\mathbf{z}))^T$, and $\mathbf{p}(\mathbf{z}) \stackrel{\text{def}}{=} (P_0(\mathbf{z}), \dots, P_{|\mathbf{M}|-1}(\mathbf{z}))^T$

Filter Bank in the Polyphase-Domain



- **Critically sampled:** $|D| = N$
- **Perfect reconstruction:** $G_p(z)H_p(z) = I_{|D|}$
- **Biorthogonal:** Perfect reconstruction + critically sampled
- **Orthogonal:** $H_p(z) = G_p^T(z^{-1})$ and $G_p(z)G_p^T(z^{-1}) = I$ (i.e. $G_p(z)$ is a **paraunitary** matrix)

A Reduction Theorem for Biorthogonal Filter Banks

Theorem 3. [ZhouD:04] *Suppose $\mathbf{G}(z)$ is an $N \times N$ matrix and $\mathbf{G}_{N-1}(z)$ is its submatrix obtained by deleting the last row of $\mathbf{G}(z)$. Suppose $\mathbf{H}(z)$ is another $N \times N$ matrix and $\mathbf{H}_{N-1}(z)$ is its submatrix obtained by deleting the last column of $\mathbf{H}(z)$.*

Then $\mathbf{G}(z)\mathbf{H}(z) = \mathbf{I}_N$ if and only if

$$\mathbf{G}_{N-1}(z)\mathbf{H}_{N-1}(z) = \mathbf{I}_{N-1}, \quad (1)$$

$$G_{N,i}(z) = \alpha(z)(-1)^{i+N} \det \mathbf{H}_{i,N-1}(z), \quad (2)$$

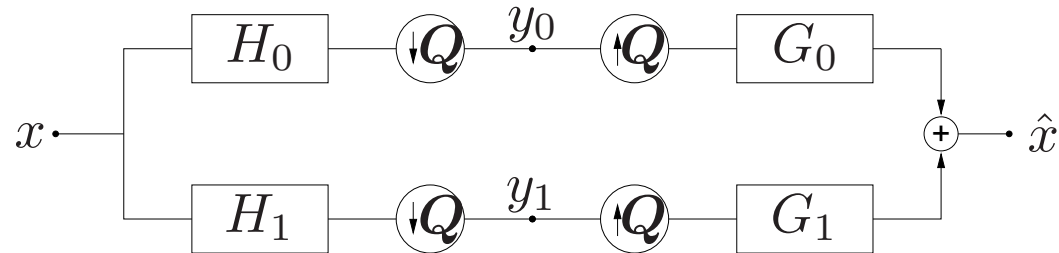
$$H_{i,N}(z) = \alpha^{-1}(z)(-1)^{i+N} \det \mathbf{G}_{N-1,i}(z), \quad (3)$$

where $\alpha(z) = \det \mathbf{G}(z)$ is an arbitrary nonzero filter, $\mathbf{H}_{i,N-1}(z)$ is the submatrix of $\mathbf{H}_{N-1}(z)$ obtained by deleting its i th row, and $\mathbf{G}_{N-1,i}(z)$ is the submatrix of $\mathbf{G}_{N-1}(z)$ obtained by deleting its i th column.

If $\mathbf{G}(z)$ and $\mathbf{H}(z)$ have FIR entries, then additionally $\alpha(z) = \alpha z^k$

Corollary 1. *A N -channel biorthogonal FIR filter bank is completely determined by its first $N - 1$ channels, and a scaled delay in the last channel.*

Example: Quincunx Filter Bank



Possible frequency partitions:



Equivalent biorthogonal condition for FIR filters:

$$H_0(z)G_0(z) + H_0(-z)G_0(-z) = 2, \quad \text{and}$$

$$H_1(z) = z^{-k}G_0(-z), \quad G_1(z) = z^k H_0(-z).$$

Filter Design Problem

Filter design amounts to solve

$$P(\mathbf{z}) + P(-\mathbf{z}) = 2, \quad (\text{easy})$$

where

$$P(\mathbf{z}) = H_0(\mathbf{z})G_0(\mathbf{z}) \quad (\text{hard})$$

Fundamental problem in multi-dimensions: no factorization theorem.

How to avoid MD factorization?

One possibility: Map 1D filters to MD filters...

Design via Mapping

1. Start with a 1D solution:

$$P^{(1D)}(z) + P^{(1D)}(-z) = 2 \quad \text{and} \quad P^{(1D)}(z) = H^{(1D)}(z)G^{(1D)}(z)$$

2. Apply a **1D to MD mapping** to each filter:

$$F^{(1D)}(z) \rightarrow F^{(MD)}(\mathbf{z}) = F^{(1D)}(M(\mathbf{z}))$$

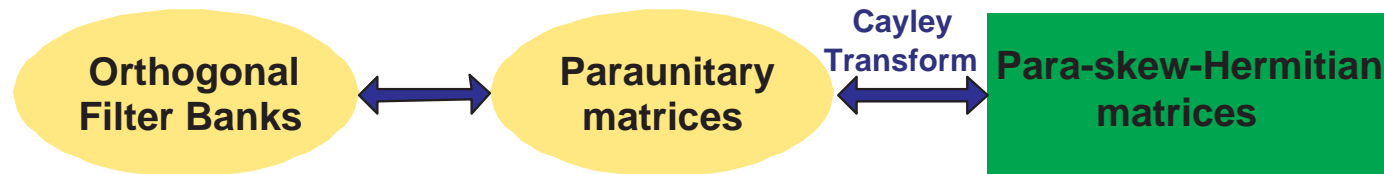
- If $M(\mathbf{z}) = -M(-\mathbf{z})$ then the perfect reconstruction condition is preserved

$$P^{(MD)}(\mathbf{z}) + P^{(MD)}(-\mathbf{z}) = P^{(1D)}(M(\mathbf{z})) + P^{(1D)}(M(-\mathbf{z})) = 2$$

- Can design the mapping to preserve and obtain other properties, like linear phase, vanishing moments, directional vanishing moments, ... (McClellan:73, TayK:93, ChenV:93, CunhaD:04)

This only works for **biorthogonal** filter banks...

MD Orthogonal FB Design using Cayley Transform



Definition. [Cayley transform] The Cayley transform of a matrix $U(z)$ is

$$H(z) = (I + U(z))^{-1}(I - U(z)).$$

Definition. $H(z)$ is a **para-skew-Hermitian (PSH)** matrix if it satisfies

$$H(z^{-1}) = -H^T(z), \quad \text{for real coefficients.}$$

Proposition 1. [ZhouDK:04] Suppose $U(z)$ is an $N \times N$ matrix. Then there exists at least one $N \times N$ diagonal matrix Λ whose diagonal entries are either 1 or -1 such that $I + \Lambda U(z)$ is nonsingular.

Theorem 4. [ZhouDK:04] The Cayley transform of a paraunitary matrix is a PSH matrix. Conversely, the Cayley transform of a PSH matrix is a paraunitary matrix.

Example: Two-Channel Filter Banks

Polyphase-domain:

$$\mathbf{U}(z) = \begin{pmatrix} U_{00}(z) & U_{01}(z) \\ U_{10}(z) & U_{11}(z) \end{pmatrix}$$

The **paraunitary** condition

$$\mathbf{U}(z)\mathbf{U}^T(z^{-1}) = \mathbf{I}$$

becomes

$$\begin{aligned} U_{00}(z)U_{00}(z^{-1}) + U_{01}(z)U_{01}(z^{-1}) &= 1, \\ U_{00}(z)U_{10}(z^{-1}) + U_{01}(z)U_{11}(z^{-1}) &= 0, \\ U_{10}(z)U_{10}(z^{-1}) + U_{11}(z)U_{11}(z^{-1}) &= 1. \end{aligned}$$

→ **nonlinear** equations

Cayley-domain:

$$\mathbf{H}(z) = \begin{pmatrix} H_{00}(z) & H_{01}(z) \\ H_{10}(z) & H_{11}(z) \end{pmatrix}.$$

The **PSH** condition

$$\mathbf{H}(z^{-1}) = -\mathbf{H}^T(z)$$

becomes

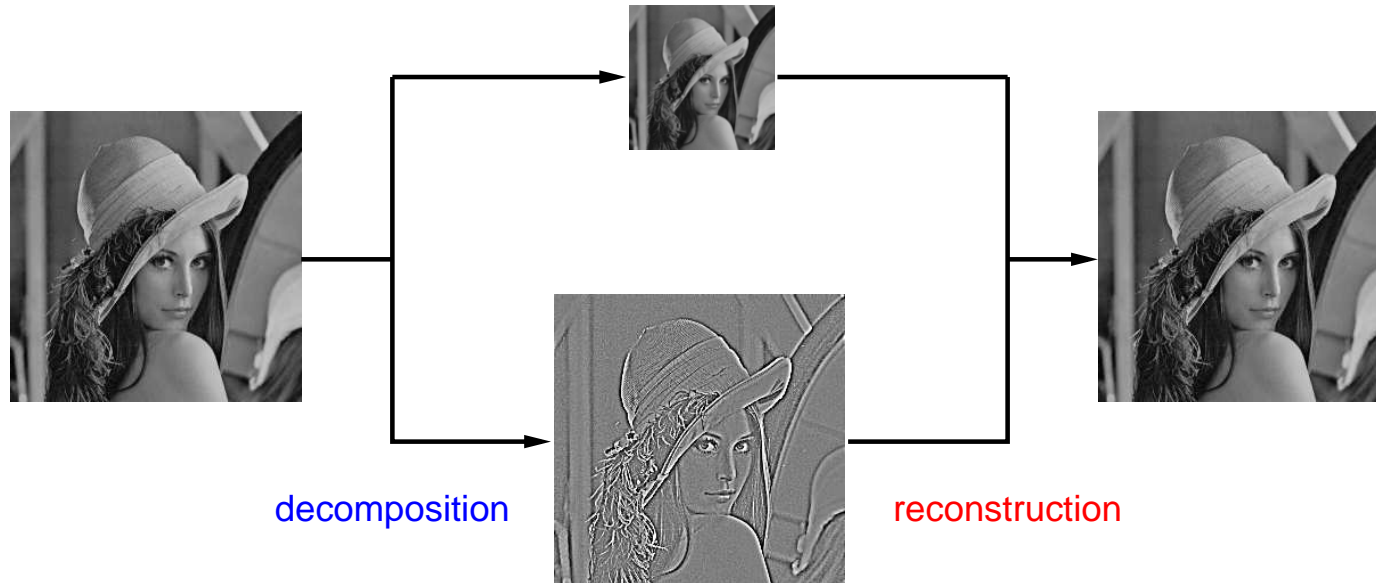
$$\begin{aligned} H_{00}(z^{-1}) &= -H_{00}(z), \\ H_{11}(z^{-1}) &= -H_{11}(z), \\ H_{01}(z^{-1}) &= -H_{10}(z). \end{aligned}$$

→ **linear** equations

Outline

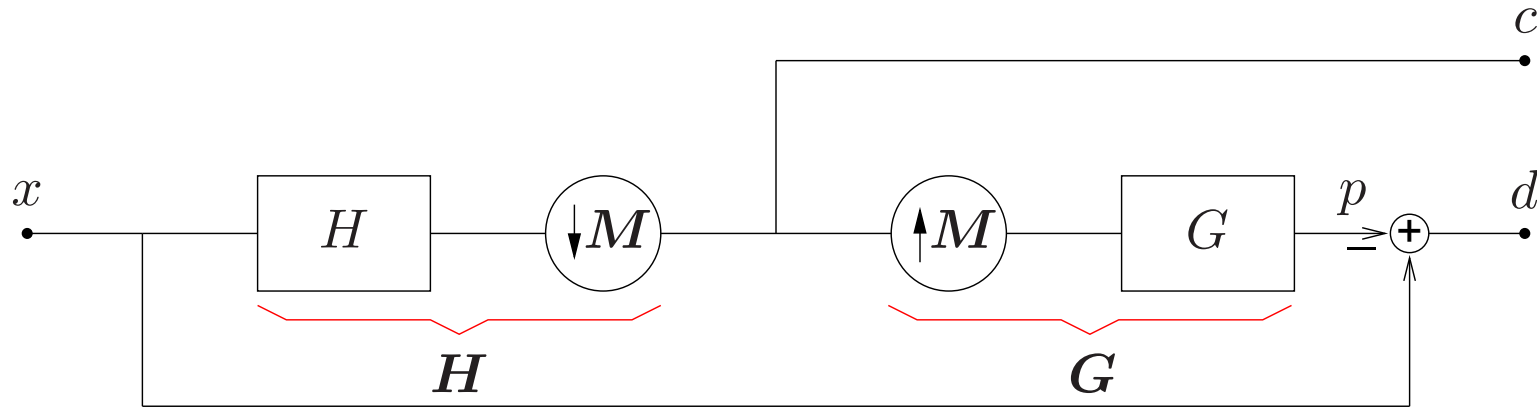
1. Multidimensional filter banks
2. Laplacian pyramid frames
 - Framing pyramids
 - New reconstruction (dual frame)
3. Iterated directional filter banks

2. Laplacian Pyramid Frames



- **Reason:** avoid “frequency scrambling” due to (\downarrow) of the HP channel.
- Laplacian pyramid as a **frame operator** \rightarrow **tight frame** exists.
- New reconstruction: efficient **filter bank** for **dual frame** (pseudo-inverse).

Decomposition in the Laplacian Pyramid



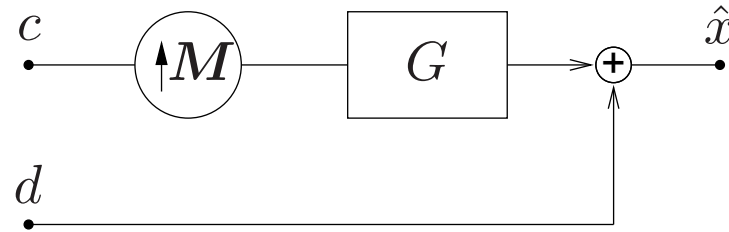
$$C(z) = H(z)x(z)$$

$$d(z) = x(z) - G(z)H(z)x(z) = (I - G(z)H(z))x(z).$$

Combining gives

$$\underbrace{\begin{pmatrix} C(z) \\ d(z) \end{pmatrix}}_{y(z)} = \underbrace{\begin{pmatrix} H(z) \\ I - G(z)H(z) \end{pmatrix}}_{A(z)} x(z).$$

Usual Reconstruction in the Laplacian Pyramid



$$\hat{x}(z) = G(z)C(z) + d(z)$$

Or,

$$\hat{x}(z) = \underbrace{\begin{pmatrix} G(z) & I \end{pmatrix}}_{S_1(z)} \underbrace{\begin{pmatrix} C(z) \\ d(z) \end{pmatrix}}_{y(z)}.$$

Note that $S_1(z)A(z) = I$ (perfect reconstruction) for any $H(z)$ and $G(z)$.

But... what about noisy pyramids: $\hat{y} = y + e$?

Frame Analysis

- LP is a **frame operator** (\mathbf{A}) with redundancy.
- It admits an **infinite number of left inverses**.
- Let \mathbf{S} be an **arbitrary** left inverse of \mathbf{A} ,

$$\hat{\mathbf{x}} = \mathbf{S}\hat{\mathbf{y}} = \mathbf{S}(\mathbf{y} + \mathbf{e}) = \mathbf{x} + \mathbf{S}\mathbf{e}.$$

- The **optimal left inverse** (minimizing $\|\mathbf{S}\|$) is the **pseudo-inverse** (or **dual frame**) of \mathbf{A} :

$$\mathbf{A}^\dagger = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T.$$

- If the noise is **white**, then among all left inverses, the pseudo-inverse minimizes the **reconstruction MSE**.

A Tight Frame Pyramid

Proposition. *The Laplacian pyramid with orthogonal filters is a **tight frame** with frame bounds equal to 1.*

Proof: Orthogonal condition means $\mathbf{G}^*(z)\mathbf{G}(z) = \mathbf{I}$ and $\mathbf{H}(z) = \mathbf{G}^*(z)$. With this, we can directly verify that

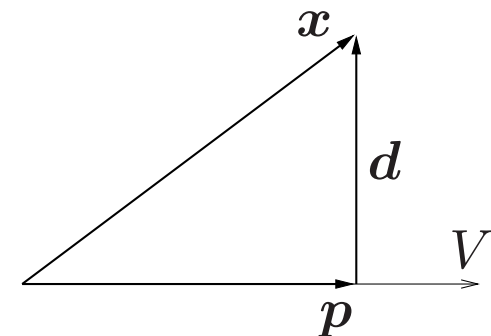
$$\mathbf{A}^*(z)\mathbf{A}(z) = \begin{pmatrix} \mathbf{H}^*(z) & \mathbf{I} - \mathbf{H}^*(z)\mathbf{G}^*(z) \end{pmatrix} \begin{pmatrix} \mathbf{H}(z) \\ \mathbf{I} - \mathbf{G}(z)\mathbf{H}(z) \end{pmatrix} = \mathbf{I}.$$

Geometrical interpretation:

$$p[n] = \sum_{k \in \mathbb{Z}^d} \underbrace{\langle x[\bullet], g[\bullet - \mathbf{M}k] \rangle}_{c[k]} g[n - \mathbf{M}k].$$

Using the *Pythagorean* theorem:

$$\|\mathbf{x}\|^2 = \|\mathbf{p}\|^2 + \|\mathbf{d}\|^2 = \|\mathbf{c}\|^2 + \|\mathbf{d}\|^2.$$

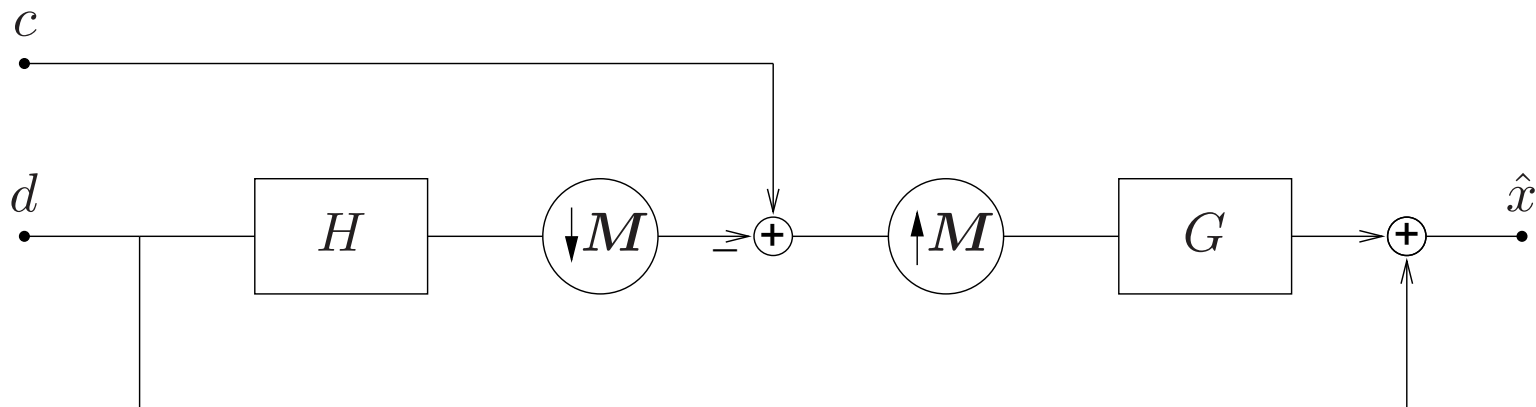


New Reconstruction for the Laplacian Pyramid

For tight frame with frame bounds 1, the pseudo-inverse of $\mathbf{A}(z)$ is

$$\mathbf{A}^\dagger(z) = \mathbf{A}^T(z) = \begin{pmatrix} \mathbf{H}(z) \\ \mathbf{I} - \mathbf{G}(z)\mathbf{G}^T(z) \end{pmatrix}^T = \begin{pmatrix} \mathbf{G}(z) & \mathbf{I} - \mathbf{G}(z)\mathbf{G}^T(z) \end{pmatrix}.$$

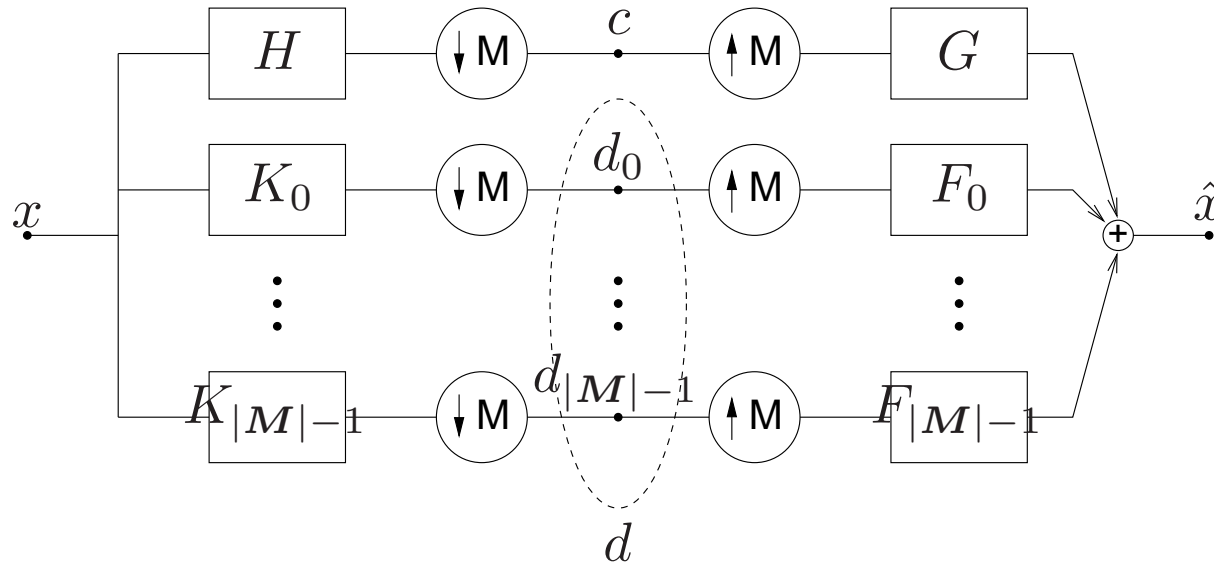
Which leads to the optimal reconstruction:



This also works for **biorthogonal** filters H and G .

Laplacian Pyramids as an Oversampled Filter Bank

From the polyphase representation of $A(z)$ and $S(z)$



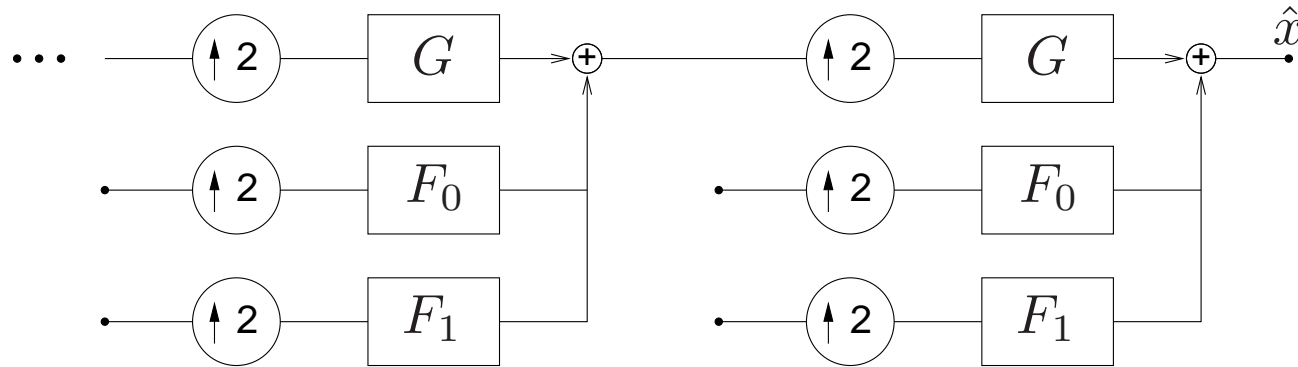
For the **usual reconstruction (REC-1)**

$$F_i^{[1]}(z) = z^{-k_i}, \quad \text{for } i = 0, \dots, |M| - 1.$$

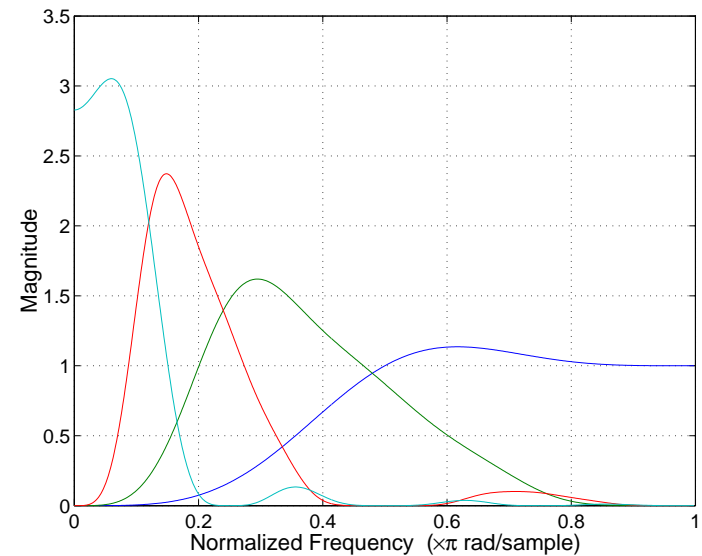
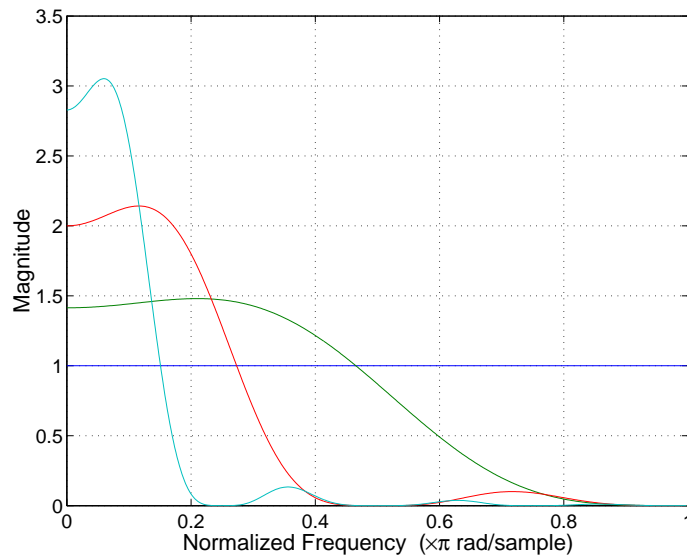
For the **new reconstruction (REC-2)**

$$F_i^{[2]}(z) = z^{-k_i} - G(z)H_i(z^M), \quad \text{for } i = 0, \dots, |M| - 1.$$

Multilevel Laplacian Pyramids

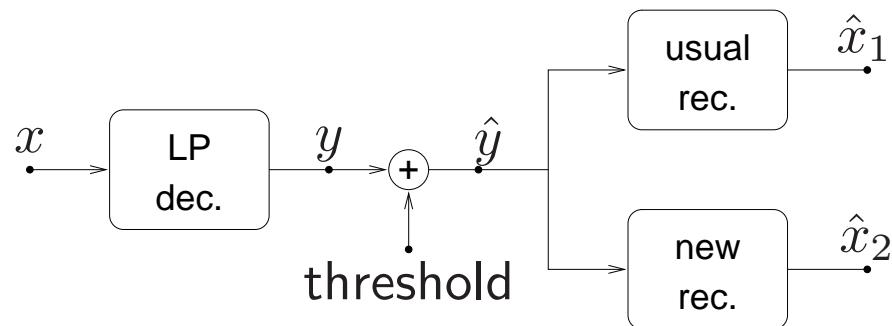


Frequency responses of equivalent synthesis filters: REC-1 and REC-2



The new reconstruction is crucial for building multiscale directional frames.

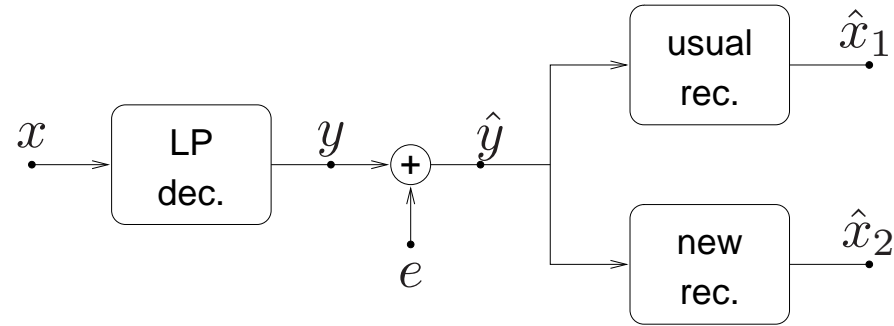
Experimental Results (1/2)



Nonlinear approximation: SNR's of the reconstructed images from the M most significant LP coefficients (after 6 levels of decomposition)

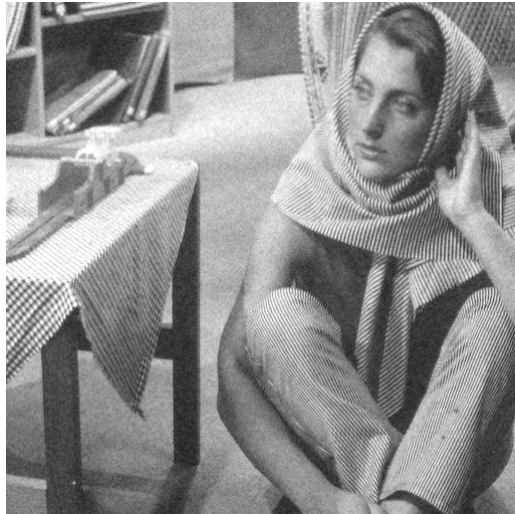
M		2^{12}	2^{14}	2^{16}
Barbara	REC-1	9.68	12.56	20.94
	REC-2	9.87	13.18	21.75
Goldhill	REC-1	12.30	15.79	21.55
	REC-2	12.60	16.23	22.19
Peppers	REC-1	15.06	20.81	26.77
	REC-2	15.62	21.33	27.32

Experimental Results (2/2)



With additive uniform white noise in $[0, 0.1]$ (**non-zero mean**)

**usual
rec.**
SNR =
6.28 dB



new rec.
SNR =
17.42 dB

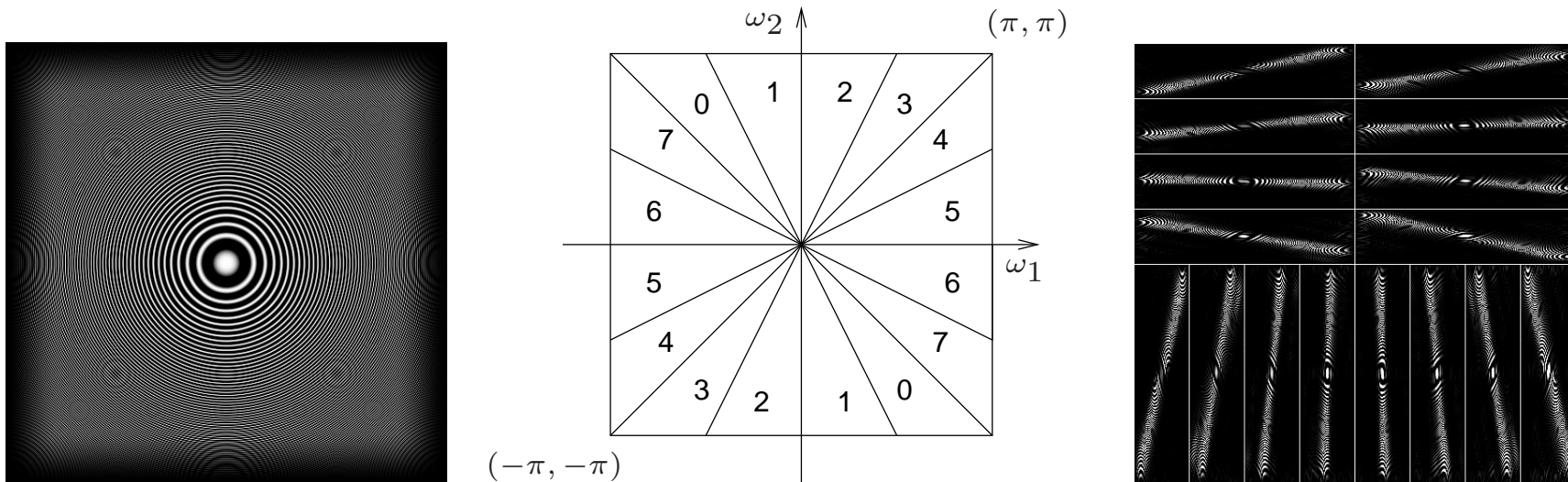


Outline

1. Multidimensional filter banks
2. Laplacian pyramid frames
3. Iterated directional filter banks
 - How to obtain directional decomposition with 2D tree-structured filter banks
 - Local directional bases

3. Iterated Directional Filter Banks

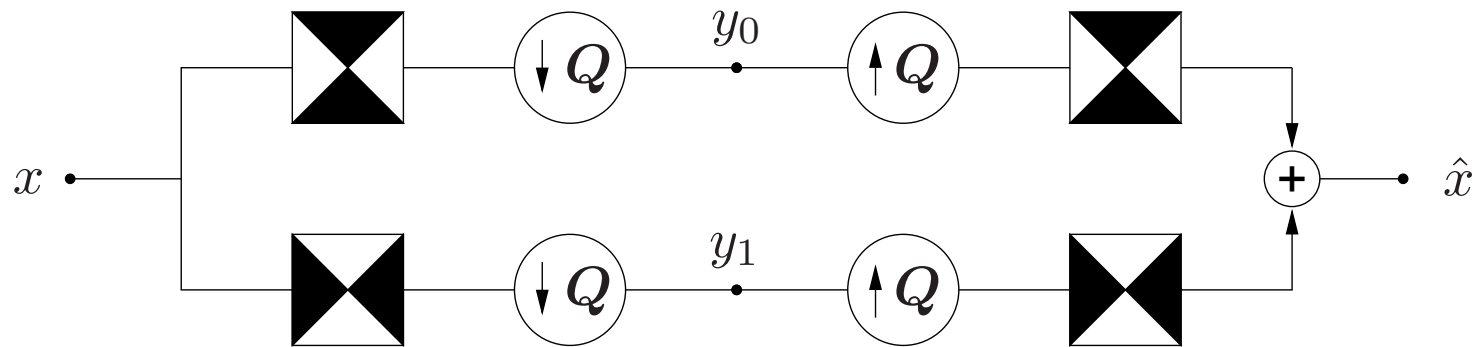
- **Feature:** division of 2-D spectrum into fine slices using **tree-structured filter banks**.



- **Background:** Bamberger and Smith ('92) cleverly used **quincunx FB's**, **modulation** and **shearing**.
- **We propose:**
 - a **simplified DFB** with fan FB's and shearing
 - use DFB to construct **directional bases**

Our Simplified DFB: Two Building Blocks

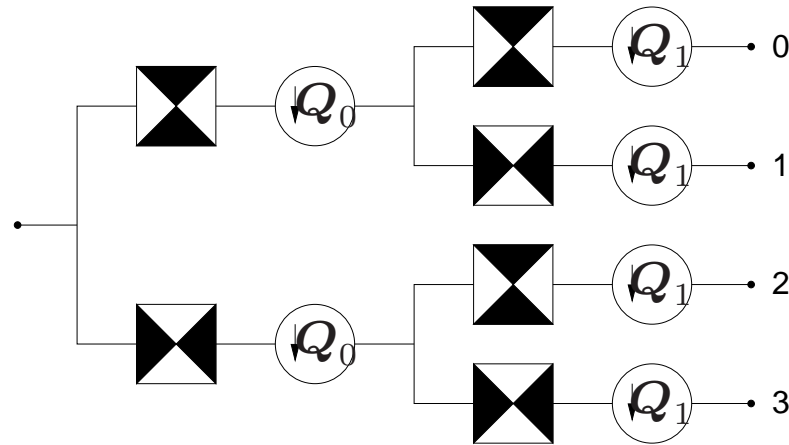
- Frequency splitting by the quincunx filter banks (Vetterli'84).



- Shearing by **resampling**



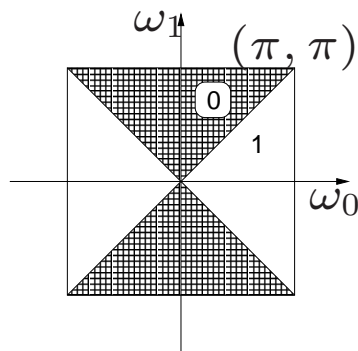
Illustration: The First Two Levels of the DFB



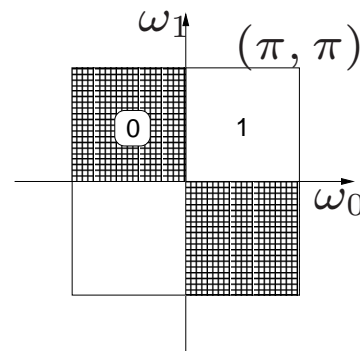
Using the **multirate identity**:

$$\downarrow M \rightarrow H(\omega) \rightarrow \equiv \rightarrow H(M^T \omega) \rightarrow \downarrow M$$

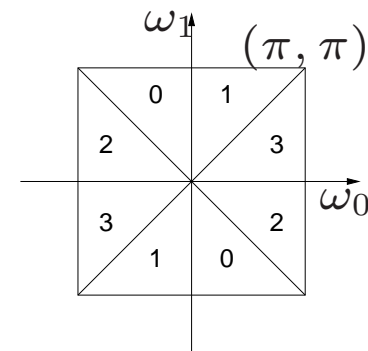
The support configurations of **equivalent filters** are:



First level

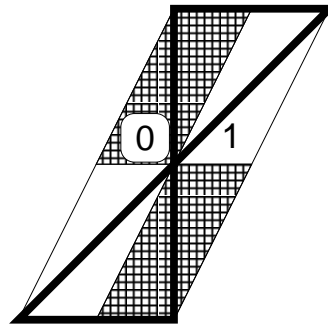
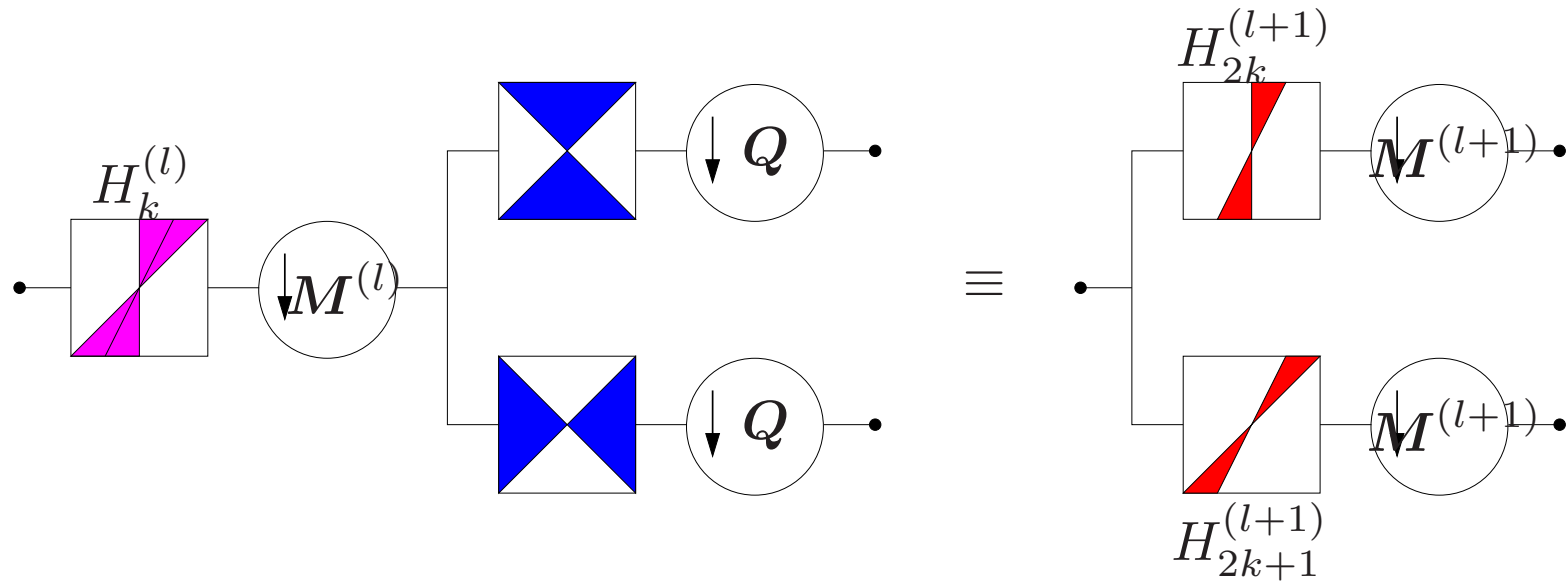


Second level



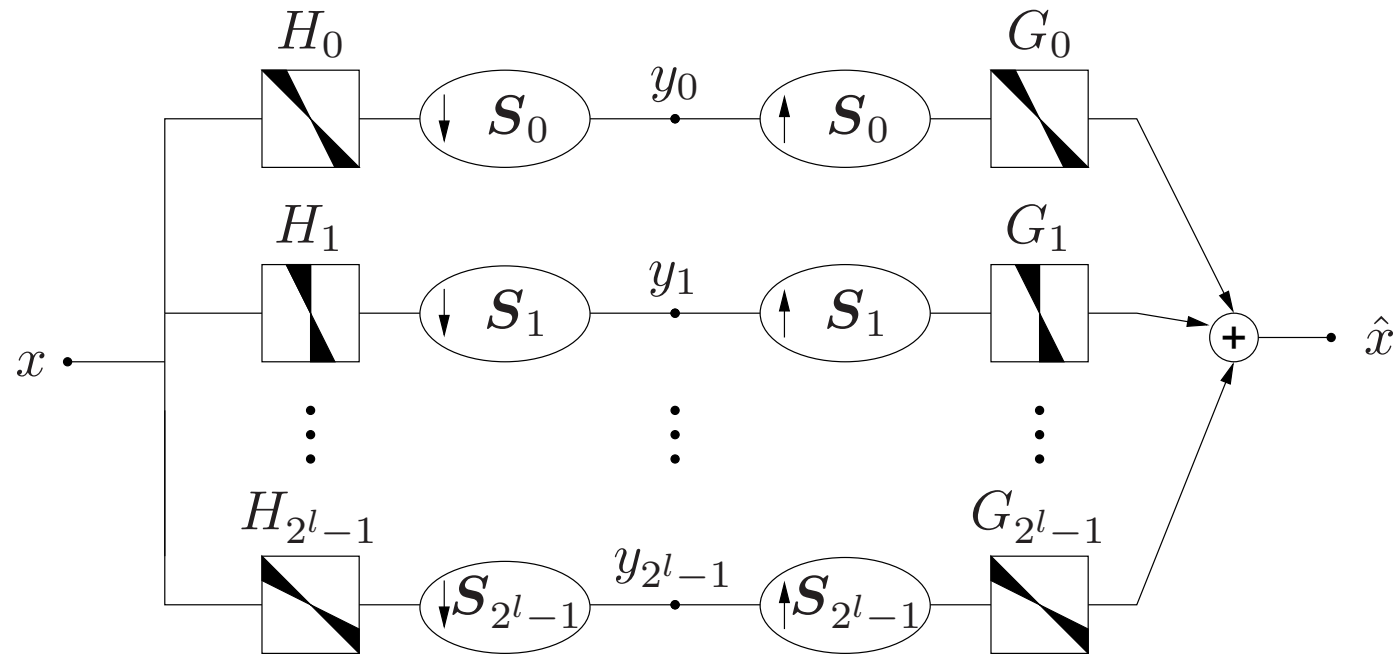
Overall filters

How Frequency is Divided into Finer Direction?



Overall sampling: $M_k^{(l)} = [2 \cdot D_i^{l-2}] \cdot R^{s_l(k)}$
 \equiv separable sampling, then shearing (doesn't change lattice)

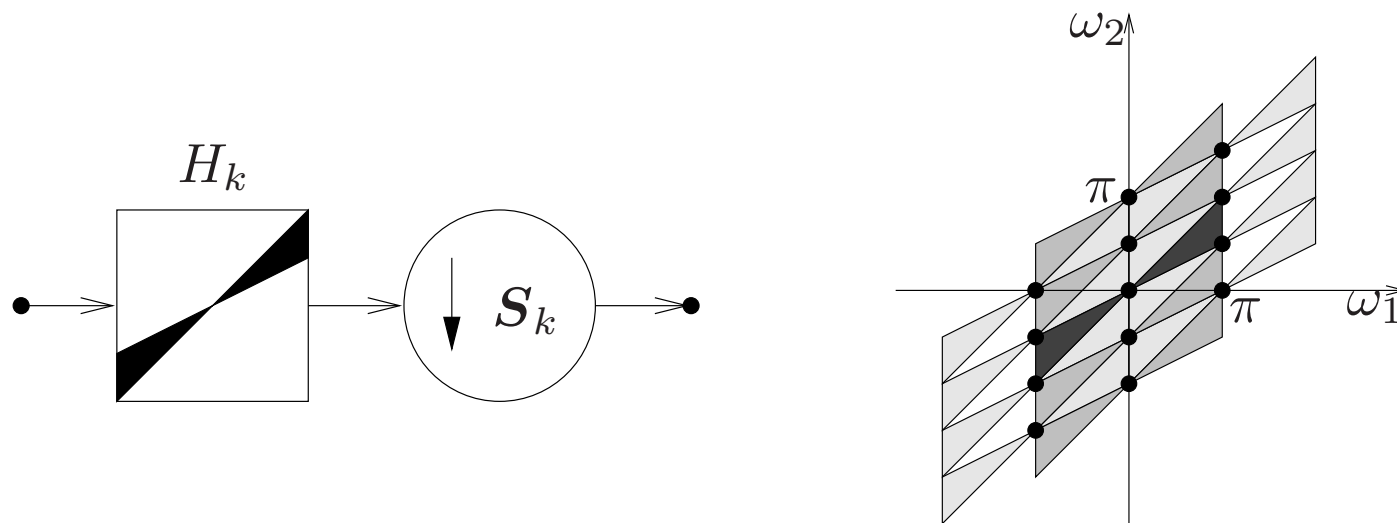
Multichannel View of the Directional Filter Bank



Use two *separable* sampling matrices:

$$S_k = \begin{cases} \begin{bmatrix} 2^{l-1} & 0 \\ 0 & 2 \end{bmatrix} & 0 \leq k < 2^{l-1} \quad (\text{"near horizontal" direction}) \\ \begin{bmatrix} 2 & 0 \\ 0 & 2^{l-1} \end{bmatrix} & 2^{l-1} \leq k < 2^l \quad (\text{"near vertical" direction}) \end{cases}$$

Why Critical Sampling Works?



Frequency tiling in the MDFB:

$$\sum_{\mathbf{m} \in \mathbb{Z}^2} \delta_{R_k}(\boldsymbol{\omega} - 2\pi \mathbf{S}_k^{-T} \mathbf{m}) = 1 \quad \text{for all } k = 0, \dots, 2^l - 1; \boldsymbol{\omega} \in \mathbb{R}^2.$$

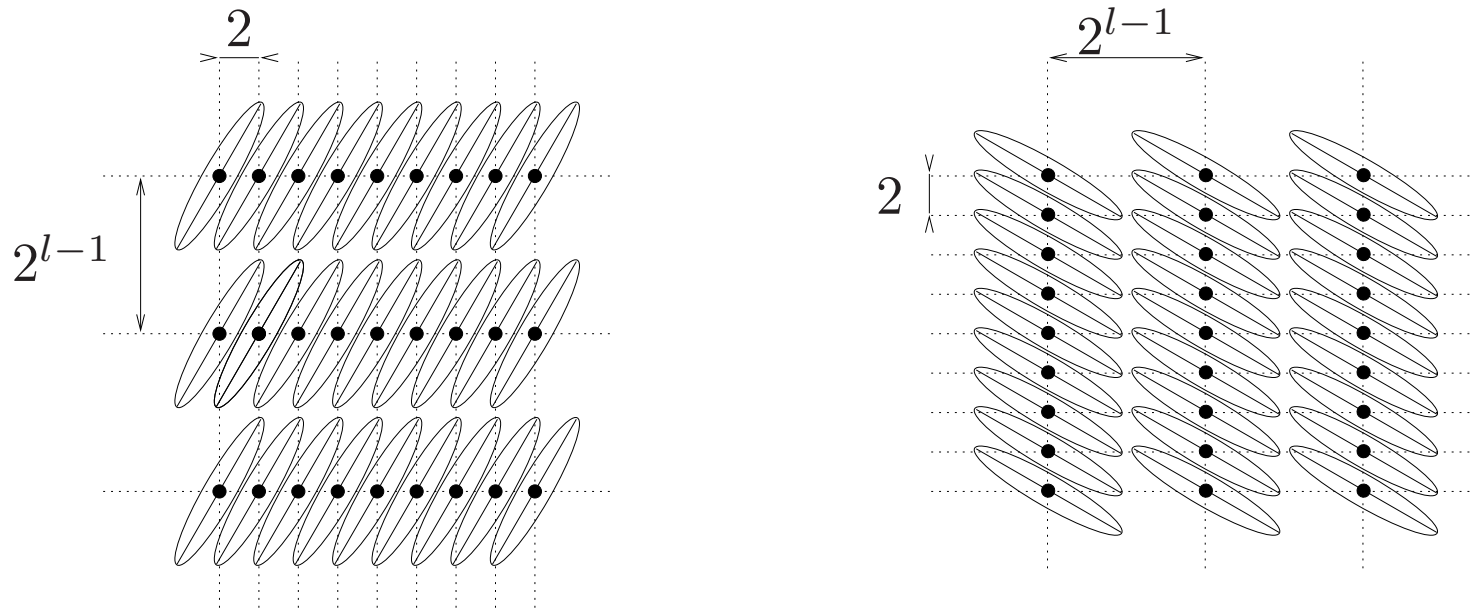
where R_k is the ideal frequency region for H_k .

General Bases from the DFB

An l -levels DFB creates a **local directional basis** of $l^2(\mathbb{Z}^2)$:

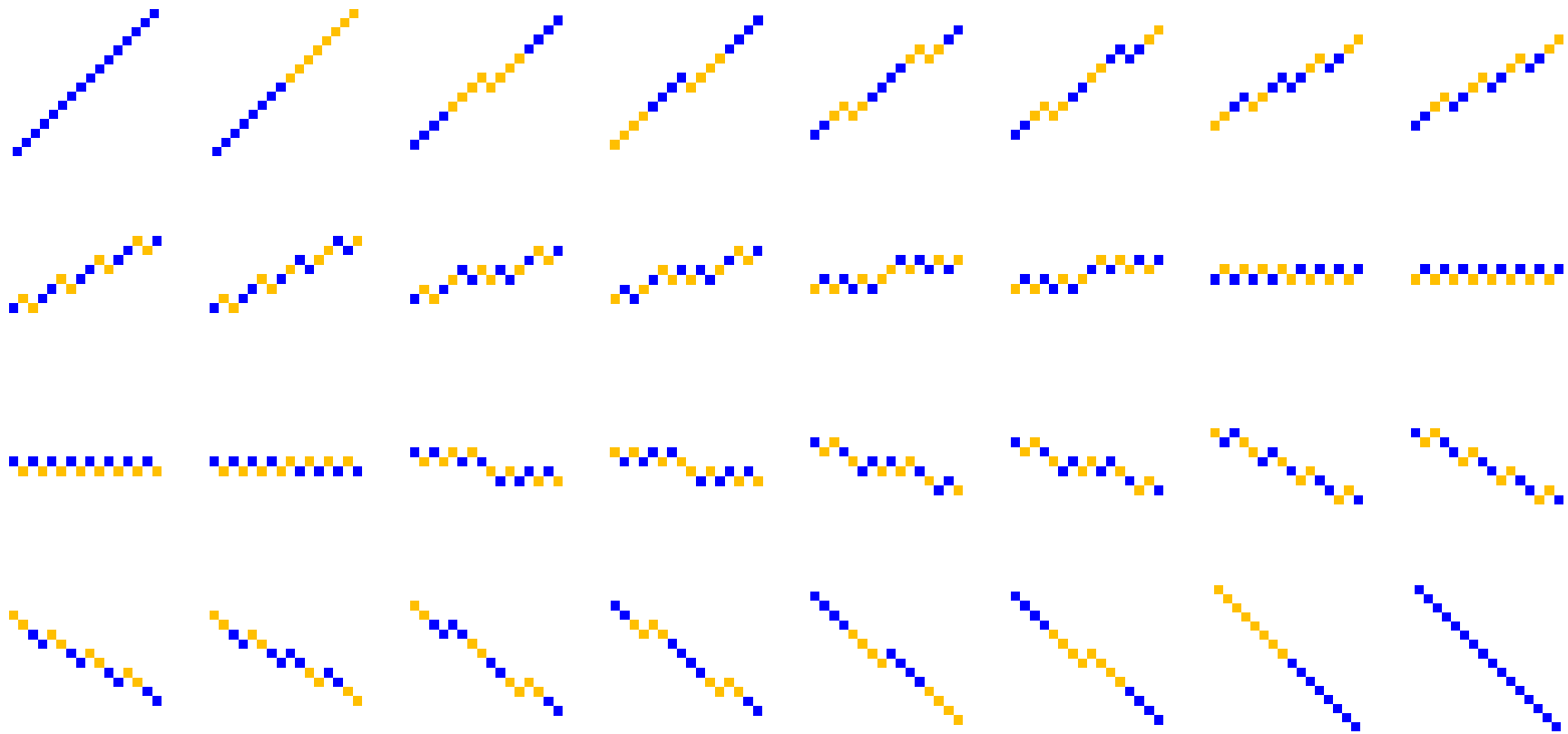
$$\left\{ g_k^{(l)}[\cdot - S_k^{(l)} n] \right\}_{0 \leq k < 2^l, n \in \mathbb{Z}^2}$$

- $G_k^{(l)}$ are directional filters:
- Sampling lattices (spatial tiling):



Example of DFB Impulse Responses

32 equivalent filters for the first half channels (basically horizontal directions) of a 6-level DFB that use the Haar filters



References

- Multidimensional filter banks
 - P. P. Vaidyanathan, *Multirate Systems and Filter Banks*, Prentice-Hall, 1993.
 - M. Vetterli and J. Kovačević, *Wavelets and Subband Coding*, Prentice-Hall, 1995.
 - J. Zhou, M. N. Do, and J. Kovačević, Multidimensional orthogonal filter bank characterization and design using the Cayley transform, *IEEE Trans. on Image Proc.*, to appear.
- Laplacian pyramid frames
 - M. N. Do and M. Vetterli, “Framing pyramids,” *IEEE Trans. on Signal Proc.*, pp. 2329-2342, Sep. 2003.
- Iterated directional filter banks
 - M. N. Do, “Directional multiresolution image representations,” Ph.D. thesis, EPFL, December 2001. (Chapter 3)
- Software and downloadable papers: www.ifp.uiuc.edu/~minhdo