Multi-directional Wavelet Transforms

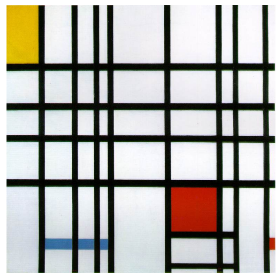
Goal: extending separable 2D wavelet transform to allow for more directions while keeping:
- simplicity of separable transforms
- low design and computational complexity
- good approximation, compression and denoising performance

Related work:
- Curvelets (Candes and Donoho)
- Contourlets (Do et al.)
- Wedgelets (Donoho, Baraniuk)
- Wedgeprints (Wakin et al.)
- Edgeprints (Dragotti and Vetterli)
- Bandelets (Mallat and Le Pennec)
- Polynomial modeling & quadtree segmentation (Shukla et al.)
- Steerable pyramid (Simoncelli)
- Directional filter banks (Bamberger and Smith)
- Complex wavelets (Kingsbury)
- Directional wavelets (Zuidwijk)

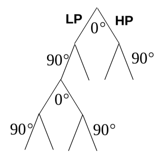
Compressing Mondrian's

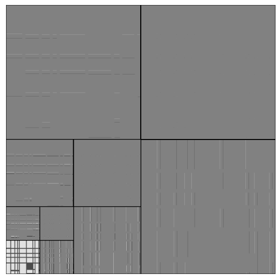
A class of images for which 2D wavelet transforms work well

$K \cdot N$



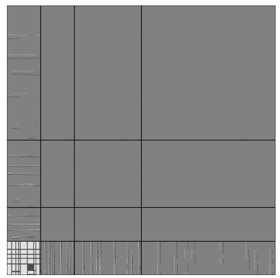
Actual Mondrian's painting





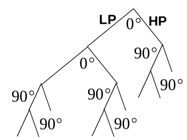
Standard wavelet transform

$\sim K \cdot N$



Fully separable wavelet transform
(repetition of transform directions)

$\sim \frac{K}{2}(\log_2 N)^2$

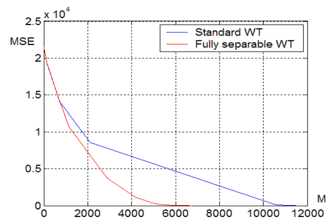


Analysis of Performance

Class: Mondrian ~ piecewise constant with straight edges along 0,90°
- Image: $N \times N$
- K edges with average length $O(N)$
- Wavelet transform: Haar with $J \sim \log_2 N$ stages

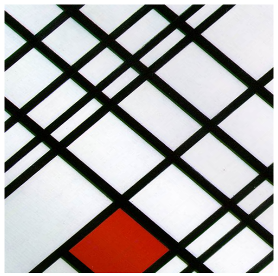
Number of nonzero coefficients:
- Standard wavelet transform: $\sim kN$
- Fully separable wavelet transform: $\sim k/2(\log_2 N)^2$

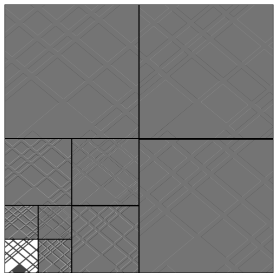
Mean-square error
- Standard wavelet transform: $e(M)=O(1/M)$
- Fully separable wavelet transform: $e(M)=O(2^{-\alpha M})$



Skewed Mondrian's

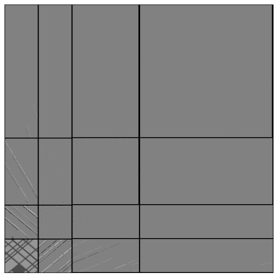
2D wavelet transform fails here





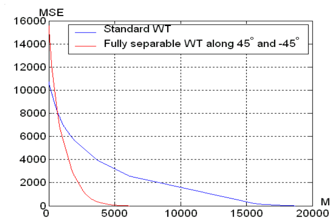
Standard wavelet transform
(filtering directions along 0°,90°)

$\sim 2KN$



Fully separable wavelet transform
(filtering directions along 45°,-45°)

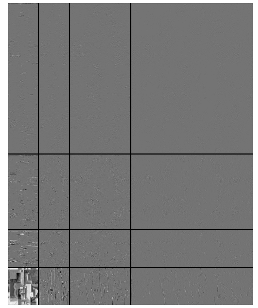
$\sim 2 \cdot \frac{K}{2}\left(\log_2 \frac{N}{2}\right)^2$



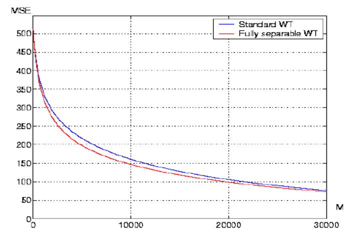
Compressing Cubists

More complex images

Standard wavelet transform
(0°,90°)



Fully separable wavelet transform
(0°,90°)

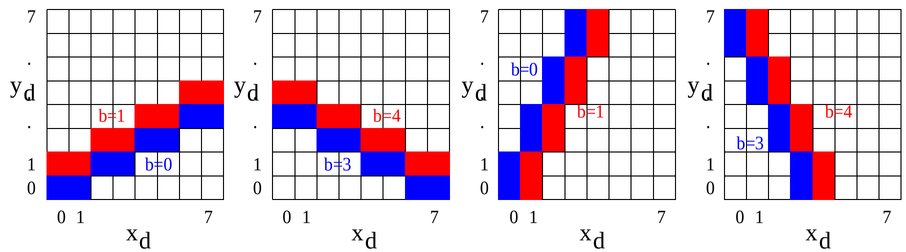


Digital Lines

Directional frames: multi-channel filtering along digital lines

From continuous to digital line (Bresenham, 1965):

- Continuous line: $y_c(x_c) = r x_c + b$
- Digital line: $y_d = \lfloor r x_d \rfloor + \lfloor b \rfloor$



$r = \frac{1}{2}$ $r = -\frac{1}{2}$ $r = 2$ $r = -2$

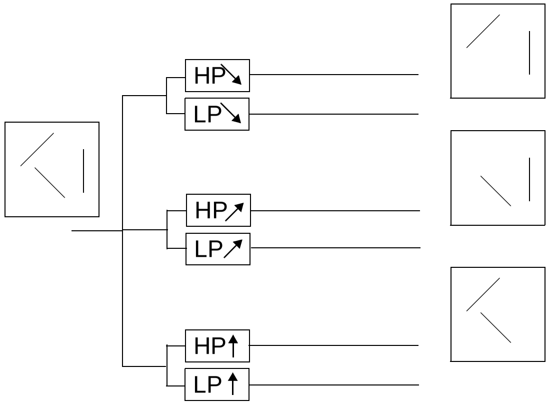
Directional Frames

Filtering along digital lines ensures:

- Complete partition of the discrete space Z^2
- Tightness of frames

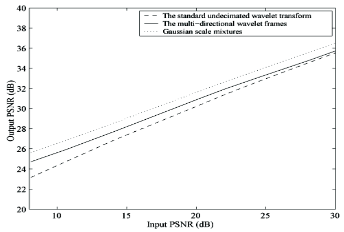
  (if 1-D filter-bank is represented by a tight frame)
- Vanishing moments along transform directions

  (imposed by high-pass filtering)



Denoising

Thresholding of directional frames



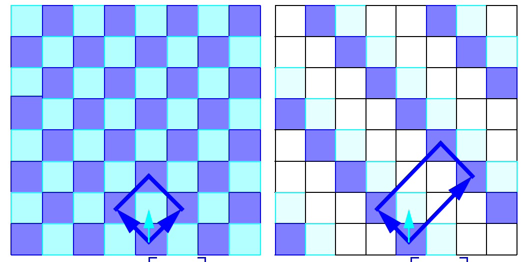
10.66dB

Standard UWT
24.75dB

Multi-directional
25.94dB

Lattice-based Filtering and Subsampling

- 1-D filtering applied along lines that belong to lattice Λ, defined by matrix M_Λ
- Processing is performed independently in each coset

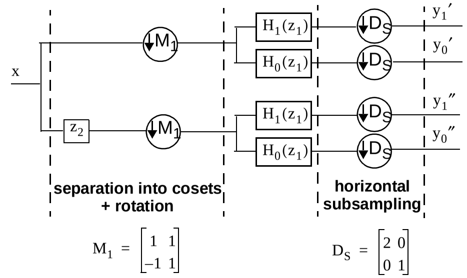


$$M_\Lambda = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad M_{\Lambda'} = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$$

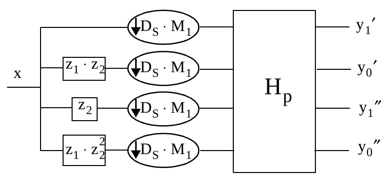
$$c_0 = \begin{bmatrix} 0 & 0 \end{bmatrix} \quad c_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Polyphase-domain Representation

1-D filter bank applied in two cosets independently

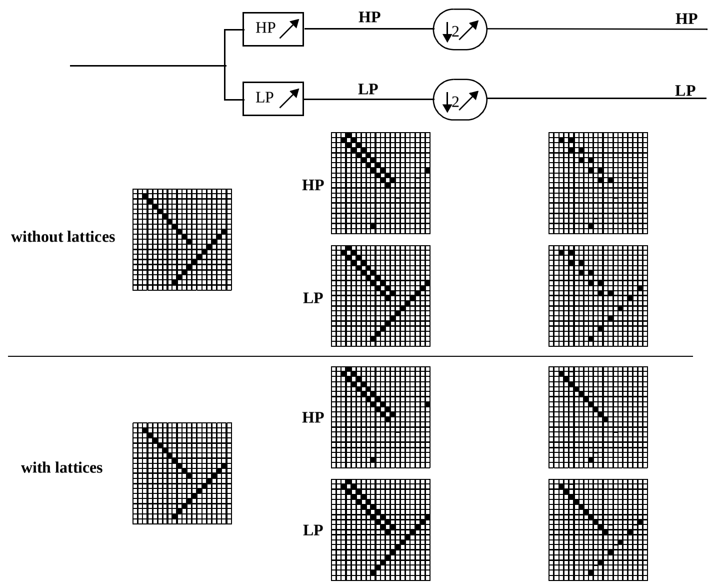


Block-diagonal analysis polyphase matrix with separable components

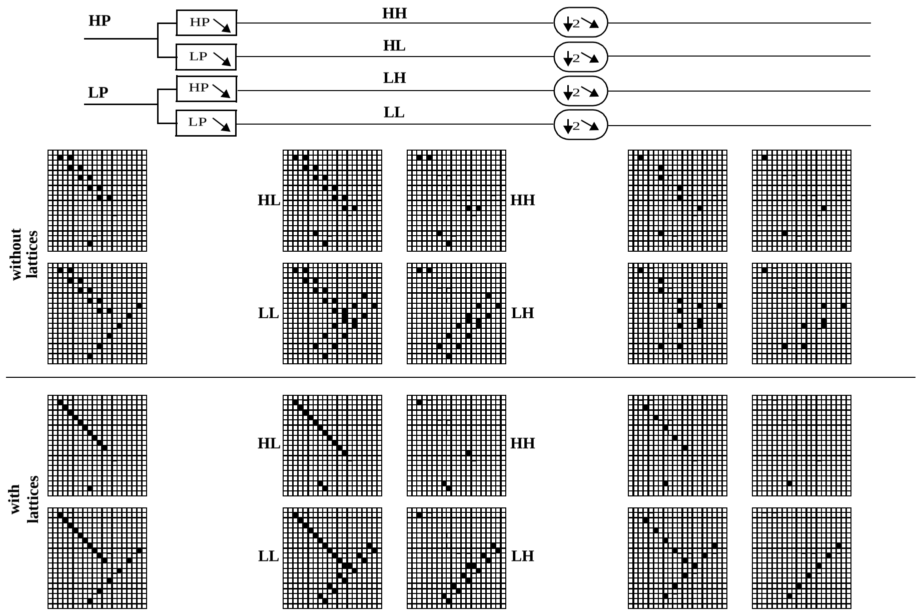


$$H_p = \begin{bmatrix} H_{00}(z_1) & H_{01}(z_1) & 0 & 0 \\ H_{10}(z_1) & H_{11}(z_1) & 0 & 0 \\ 0 & 0 & H_{00}(z_1) & H_{01}(z_1) \\ 0 & 0 & H_{10}(z_1) & H_{11}(z_1) \end{bmatrix}$$

Directional Transform: Digital Lines vs. Lattices



Iterated Filtering



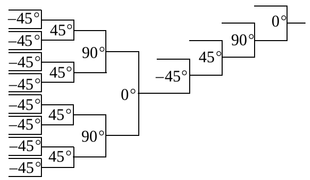
Iterated Filtering (cont'd)

Advantages of lattice-based filtering:

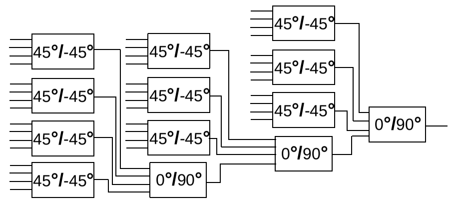
- Systematic method for iterated filtering
  (no clear rule in digital lines-based method!)
- Order of number of nonzero transform coefficients:
  $\sim kN$ (digital lines)
  $\sim \lambda k(\log_2 N/\lambda)^2$ (lattices)!

Directionlets: bases constructed by multi-directional WT

- Directionlets depend on order of transform directions
- Two examples: cyclic and two-directional synthesis constructions



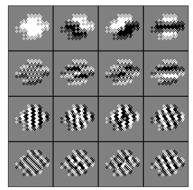
Cyclic construction with 4 directions



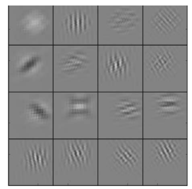
Two-directional construction with 4 directions

Basis Functions

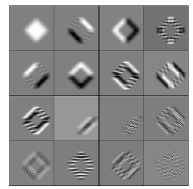
Cyclic construction has a trouble with regularity



Haar
orthogonal
easy to invert
non-smooth

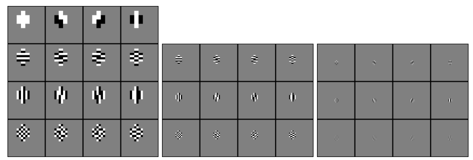


"9-7"
biorthogonal
easy to invert
smoother

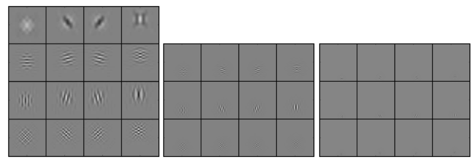


extended Haar
biorthogonal
hard to invert
smooth (regular)

Two-directional construction achieves both multi-directionality and regularity!



Haar



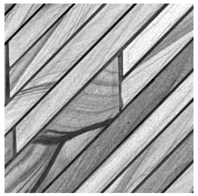
"9-7"

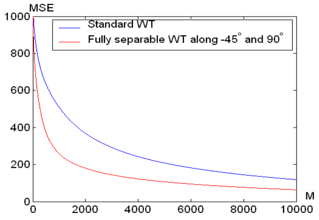
NOTICE: although functions look non-separable, only 1-D filtering is taking place here!

Compression

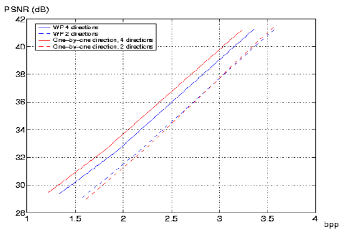
Modified SFQ (Xiong, 1996): Multi-directional SFQ

- Results are based on R-D estimation

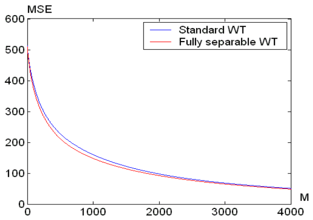




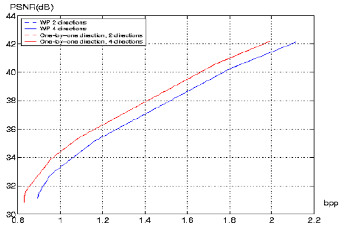
NLA



Compression



NLA

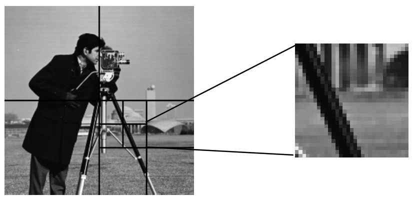


Compression

Ongoing Work

Compression

- Images are non-stationary signals, thus directionality is local property
- **Quadtree segmentation** allows for better matching between transform and dominant directions



Denoising: locally adaptive statistical model

Optimal estimation of original coefficients enforcing three types of coherence in images:

- across scales - multi-scale transform
- across space - geometrical contours
- across directions - weighting directional subbands that match dominant directions