Contourlets: Construction and Properties

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Outline

1. Motivation

2. Discrete-domain construction using filter banks

3. Contourlets and directional multiresolution analysis

4. Contourlet approximation

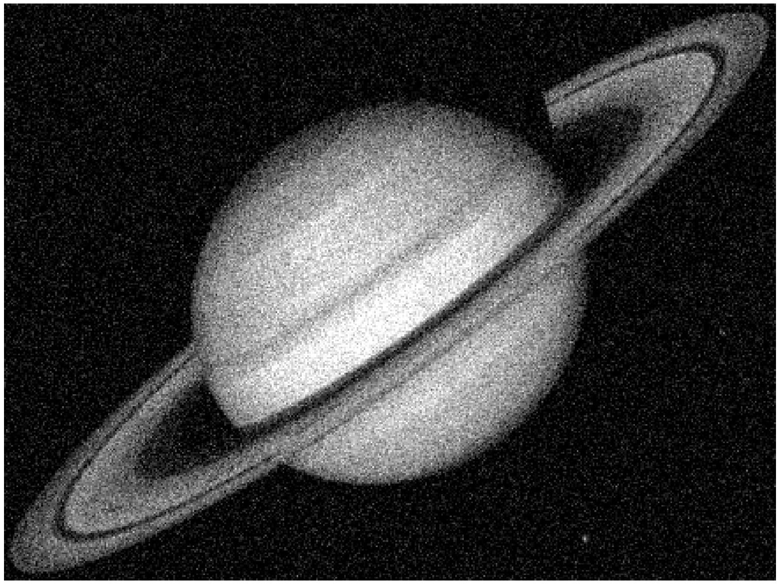
5. Contourlet filter design with directional vanishing moments

What Do Image Processors Do for Living?

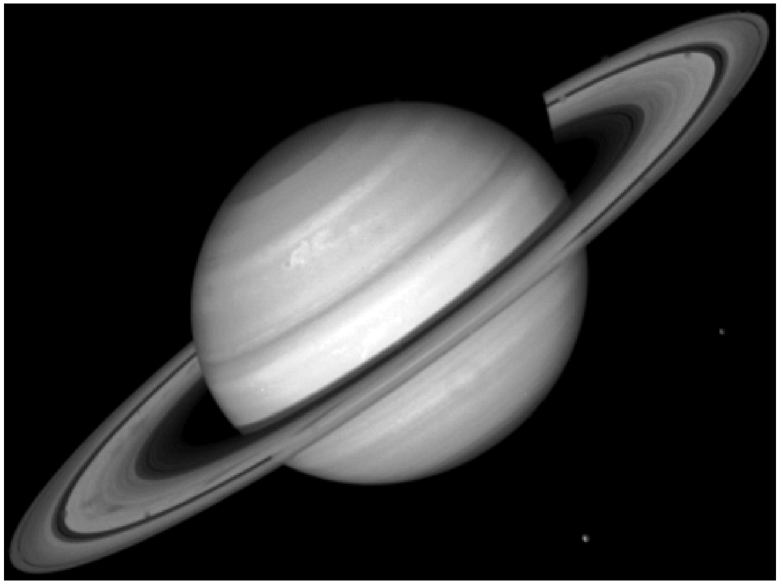
Compression: At 158:1 compression ratio...

What Do Image Processors Do for Living?

Denoising (restoration/filtering)



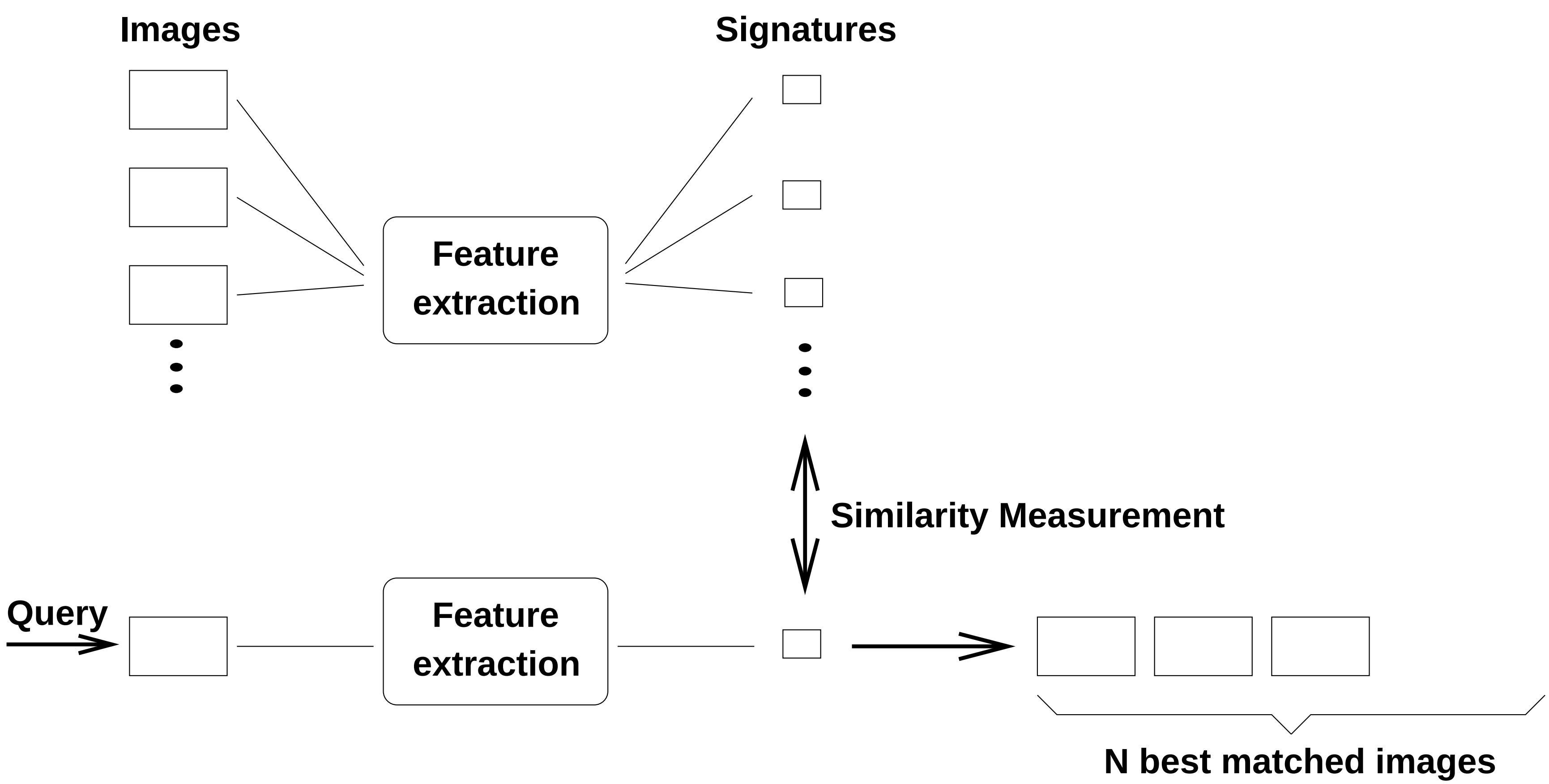
Noisy image



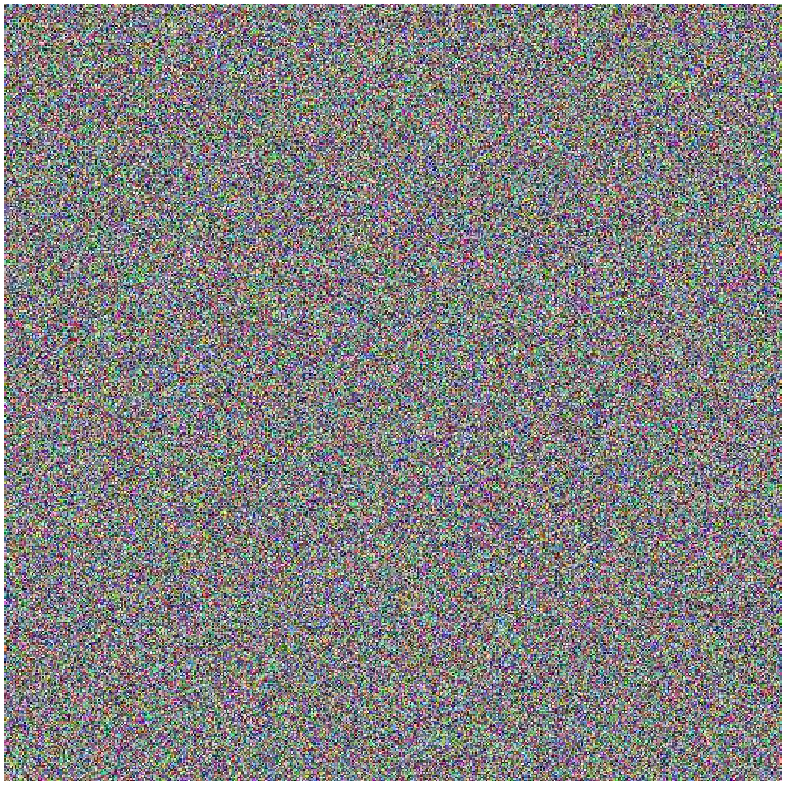
Clean image

What Do Image Processors Do for Living?

Feature extraction (e.g. for content-based image retrieval)



Fundamental Question: Parsimonious Representation of Visual Information



A randomly generated image

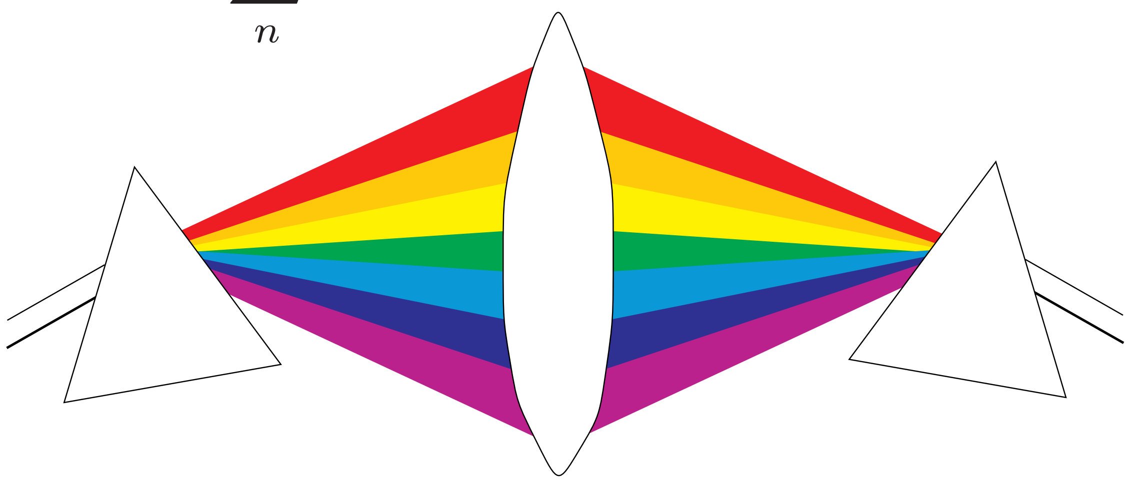
A natural image

Natural images live in a very tiny bit of the huge "image space" (e.g. $\mathbb{R}^{512\times512\times3}$)

Mathematical Foundation: Sparse Representations

Fourier, Wavelets... $=$ construction of bases for signal expansions:

$$f = \sum_n c_n \psi_n, \quad \text{where} \quad c_n = \langle f, \psi_n \rangle.$$

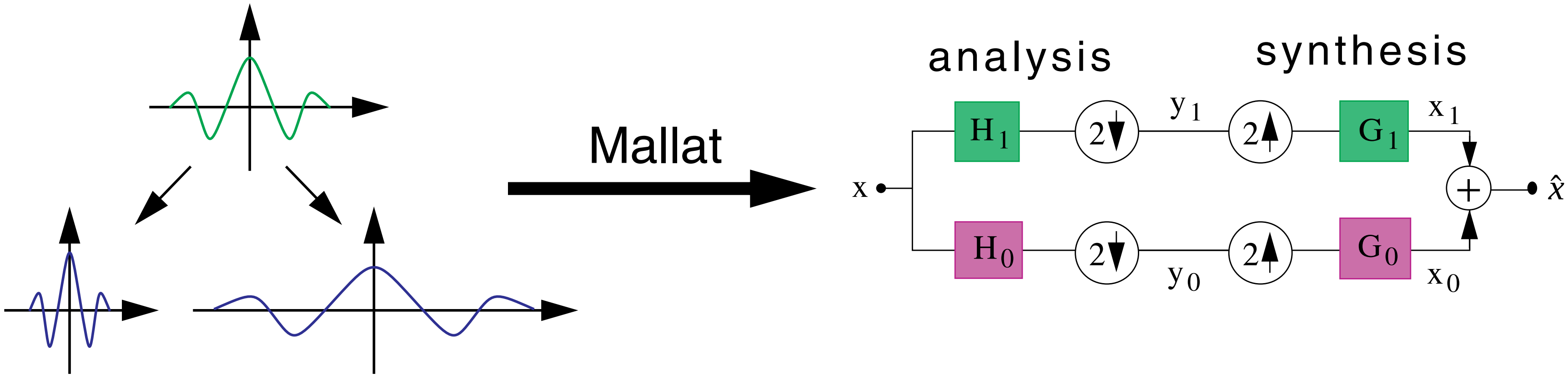


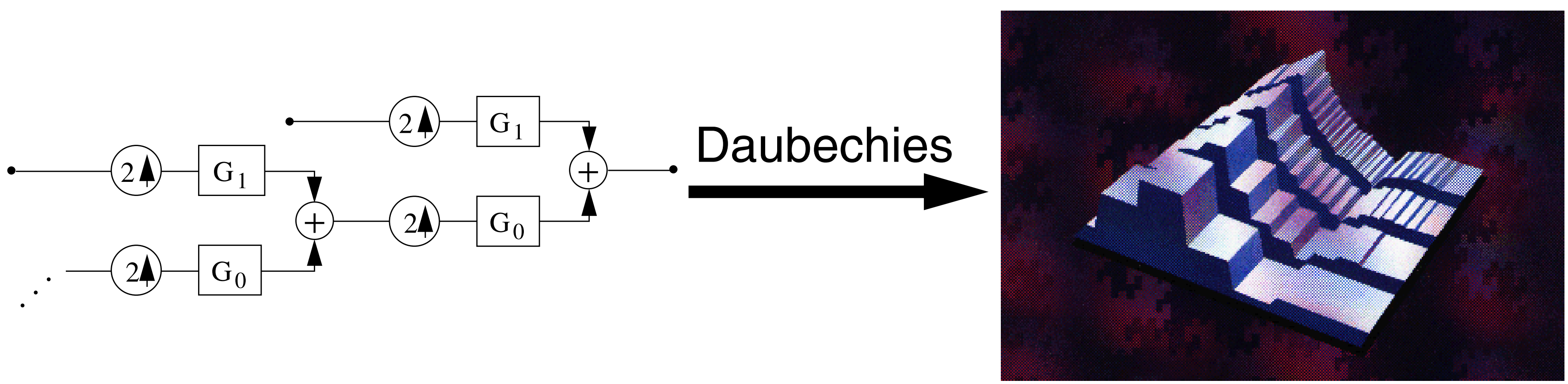
Non-linear approximation:

$$\hat{f}_M = \sum_{n \in I_M} c_n \psi_n, \qquad \text{where } I_M : \text{indexes of best } M \text{ components.}$$

Sparse representation: How fast $\|f - \hat{f}_M\| \to 0$ as $M \to \infty$.

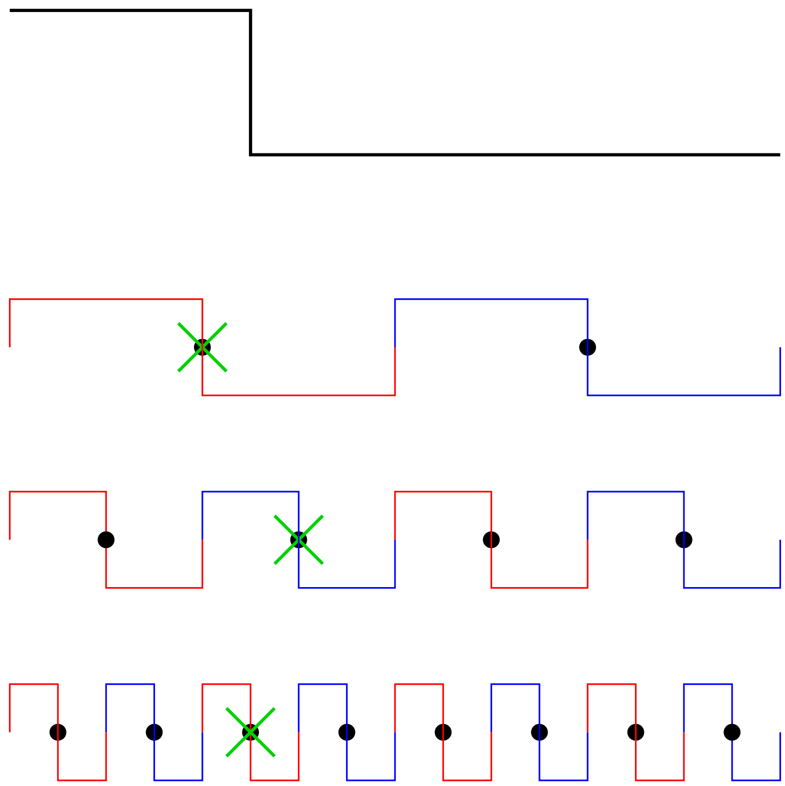
Wavelets and Filter Banks





The Success of Wavelets

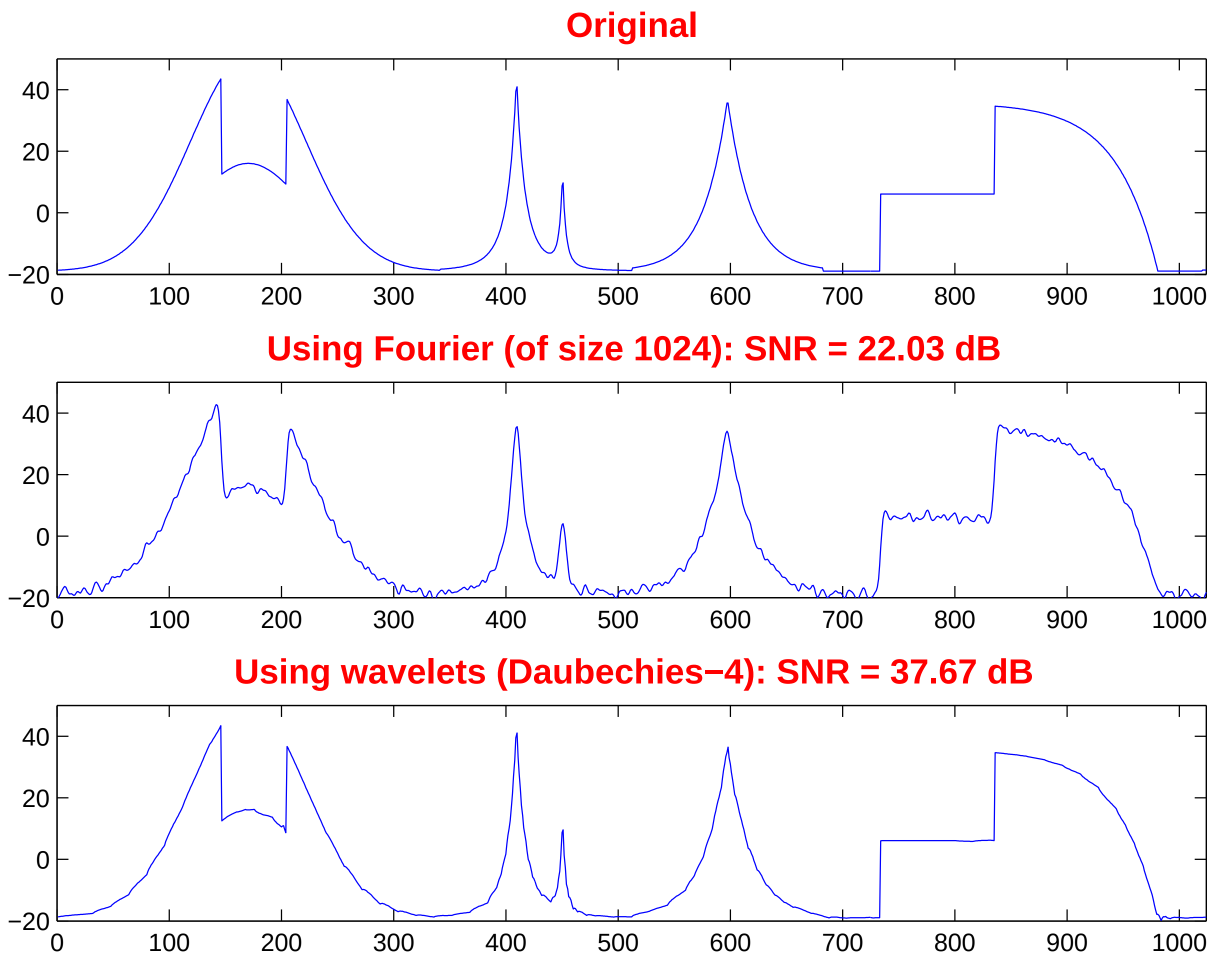
- Wavelets provide a sparse representation for piecewise smooth signals.



- Multiresolution, tree structures, fast transforms and algorithms, etc.
- Unifying theory ⇒ fruitful interaction between different fields.

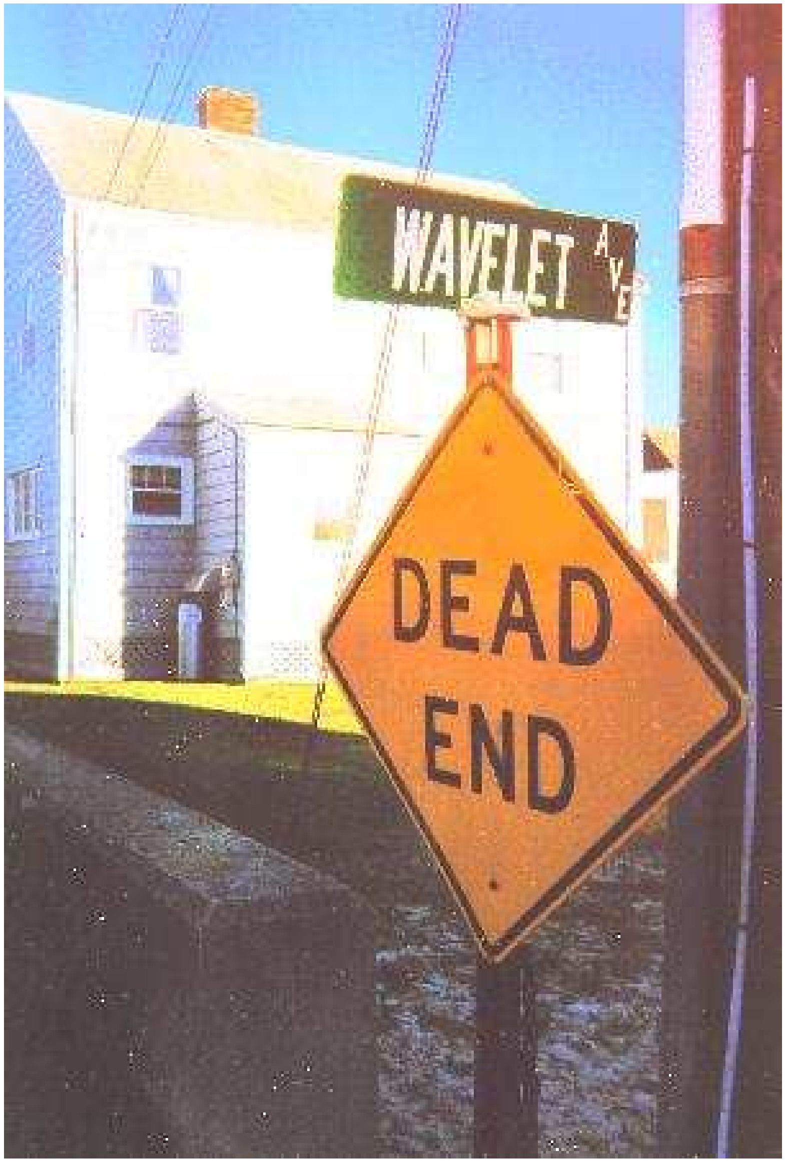
Fourier vs. Wavelets

Non-linear approximation: $N = 1024$ data samples; keep $M = 128$ coefficients

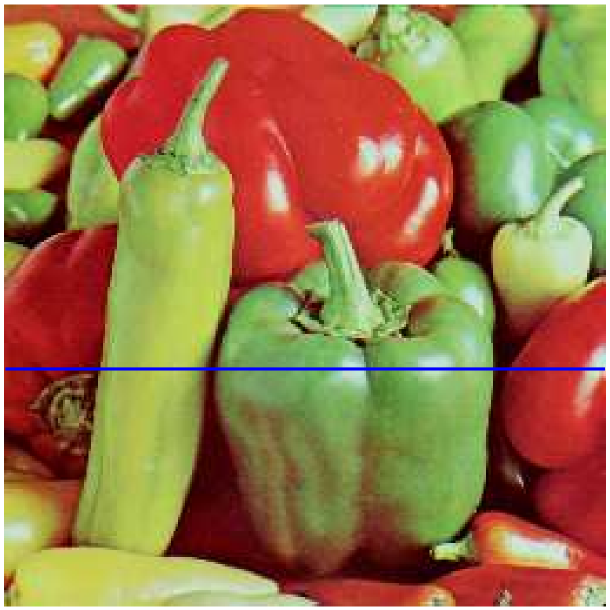


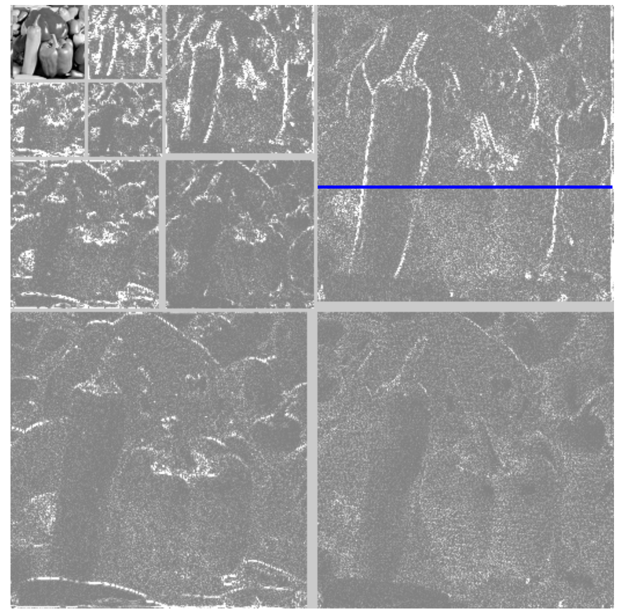
Approximation movie!

Is This the End of the Story?



Wavelets in 2-D



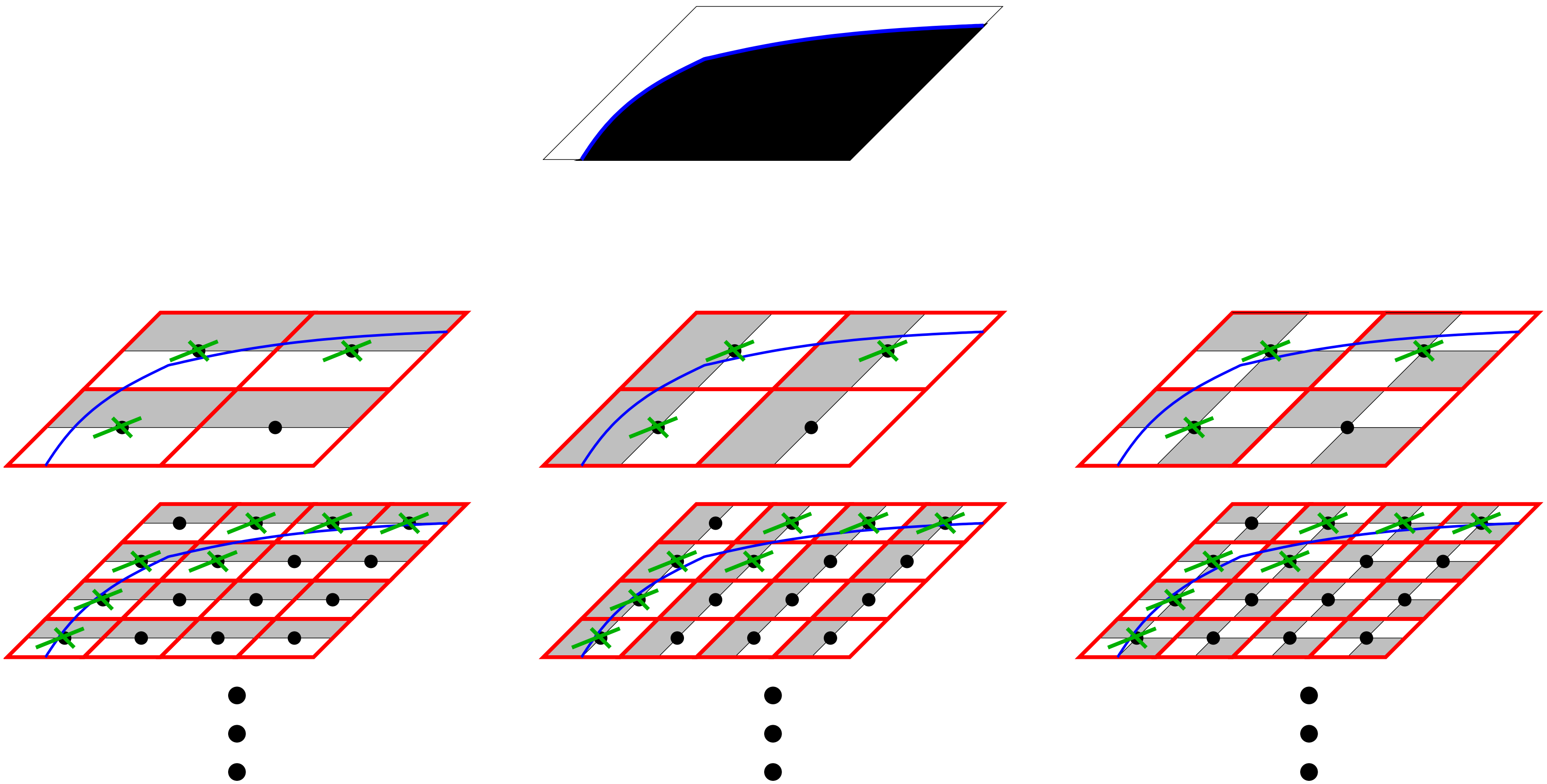


- **In 1-D:** Wavelets are well adapted to abrupt changes or singularities.

- **In 2-D:** Separable wavelets are well adapted to *point*-singularities (only).

  But, there are (mostly) *line*- and *curve*-singularities...

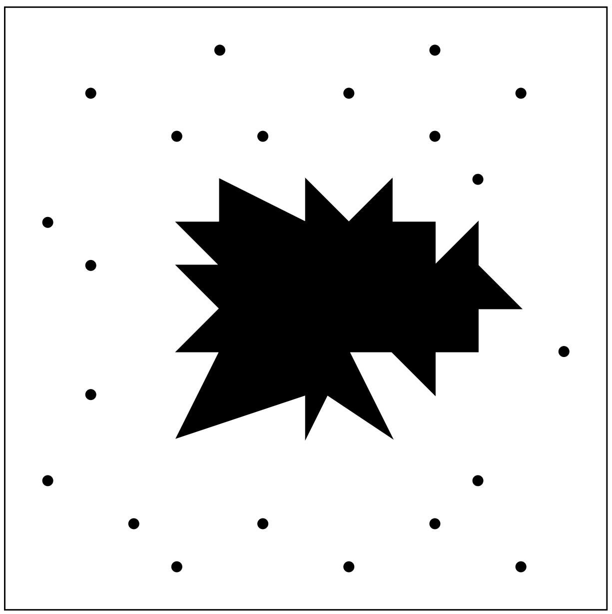
The Failure of Wavelets

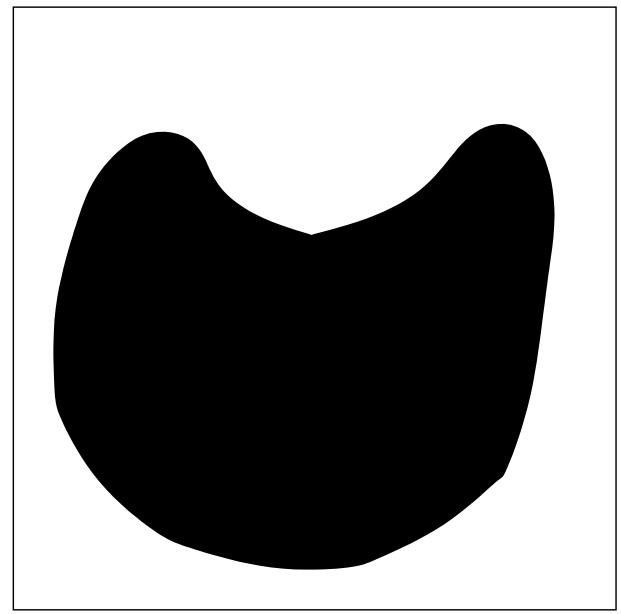


Wavelets fail to capture the geometrical regularity in images and multidimensional data.

Edges vs. Contours

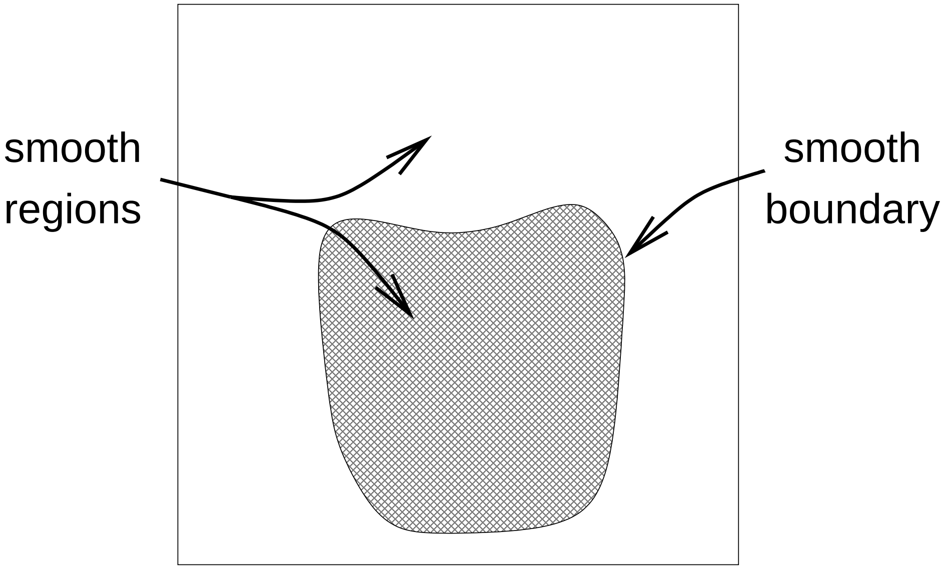
Wavelets (with nonlinear approximations) cannot "see" the difference between these two images.





- **Edges**: image points with discontinuity
- **Contours**: edges with localized and regular direction [Zucker et al.]

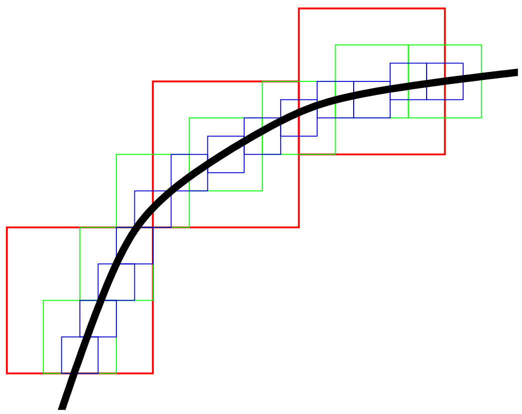
Goal: Efficient Representation for Typical Images with Smooth Contours



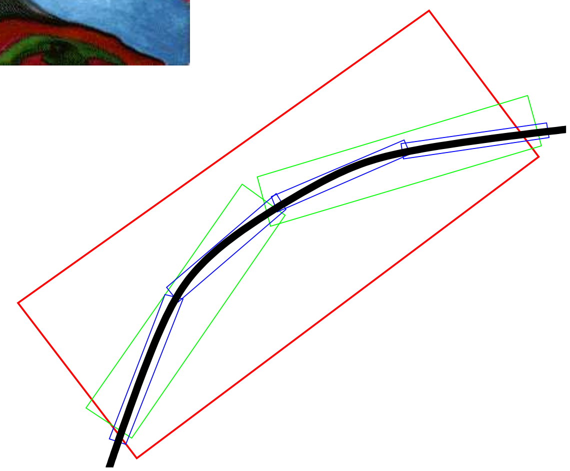
Goal: Exploring the intrinsic geometrical structure in natural images.

$\Rightarrow$ Action is at the edges!

Wavelet vs. New Scheme



Wavelets



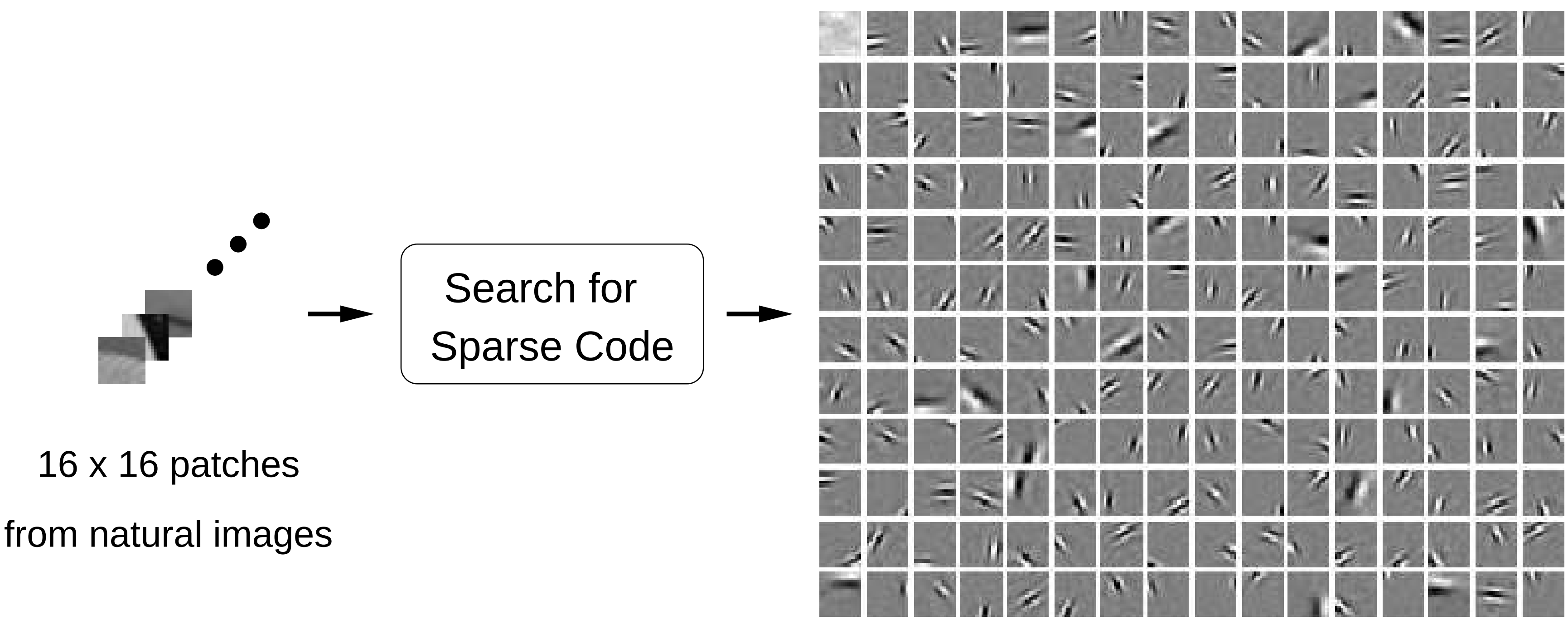
New scheme

For images:

- Wavelet scheme... see edges but not smooth contours.
- New scheme... requires challenging non-separable constructions.

And What The Nature Tells Us...

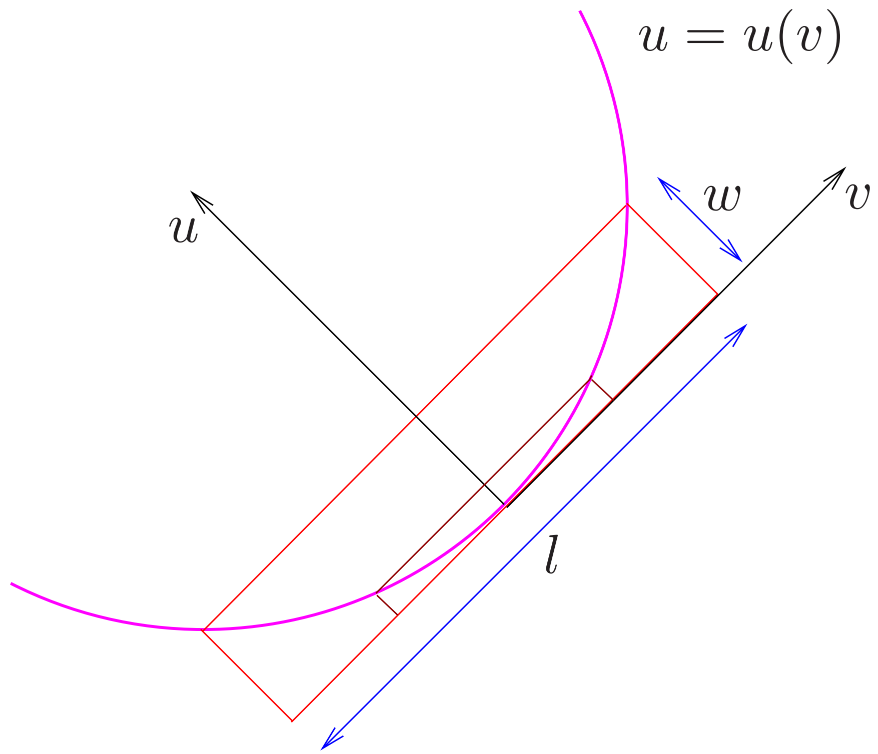
- **Human visual system**:
  - Extremely efficient: 10^7 bits $\longrightarrow$ 20-40 bits (per second).
  - Receptive fields are characterized as localized, multiscale and oriented.
- Sparse components of natural images (Olshausen and Field, 1996):



Recent Breakthrough from Harmonic Analysis: Curvelets [Candès and Donoho, 1999]

- Optimal representation for functions in $\mathbb{R}^2$ with curved singularities.
- **Key idea:** parabolic scaling relation for C^2 curves:

$$width \propto length^2$$



“Wish List” for New Image Representations

- Multiresolution ... successive refinement
- Localization ... both space and frequency
- Critical sampling ... correct joint sampling
- Directionality ... more directions
- Anisotropy ... more shapes

Our emphasis is on discrete framework that leads to algorithmic implementations.

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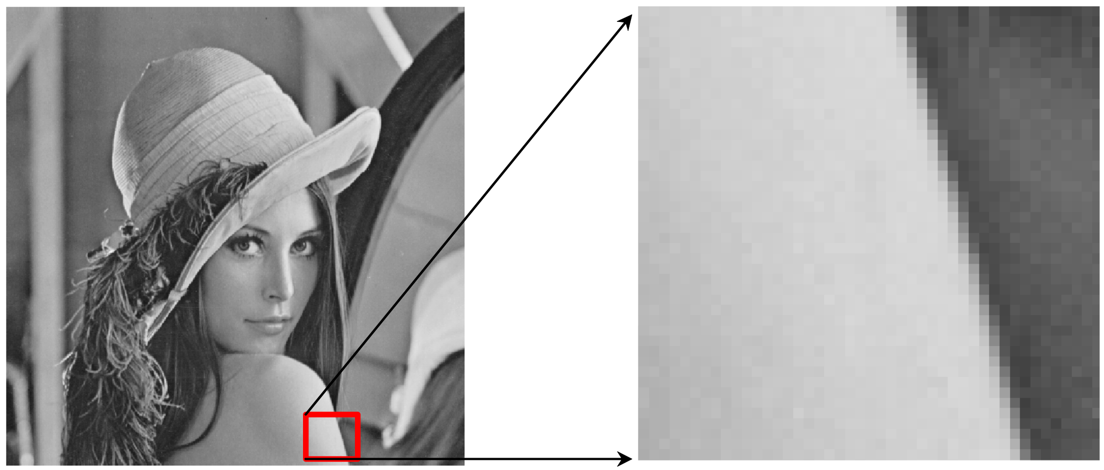
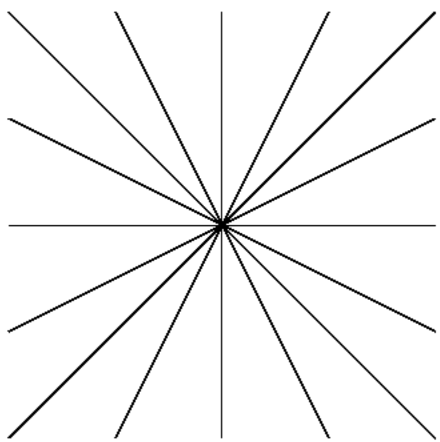
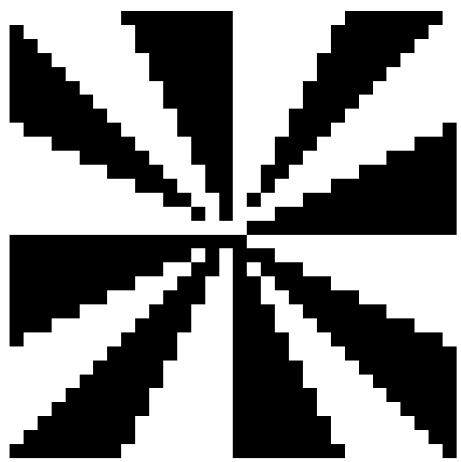
Challenge: Being Digital!

Pixelization:



Digital directions:





Proposed Computational Framework: Contourlets

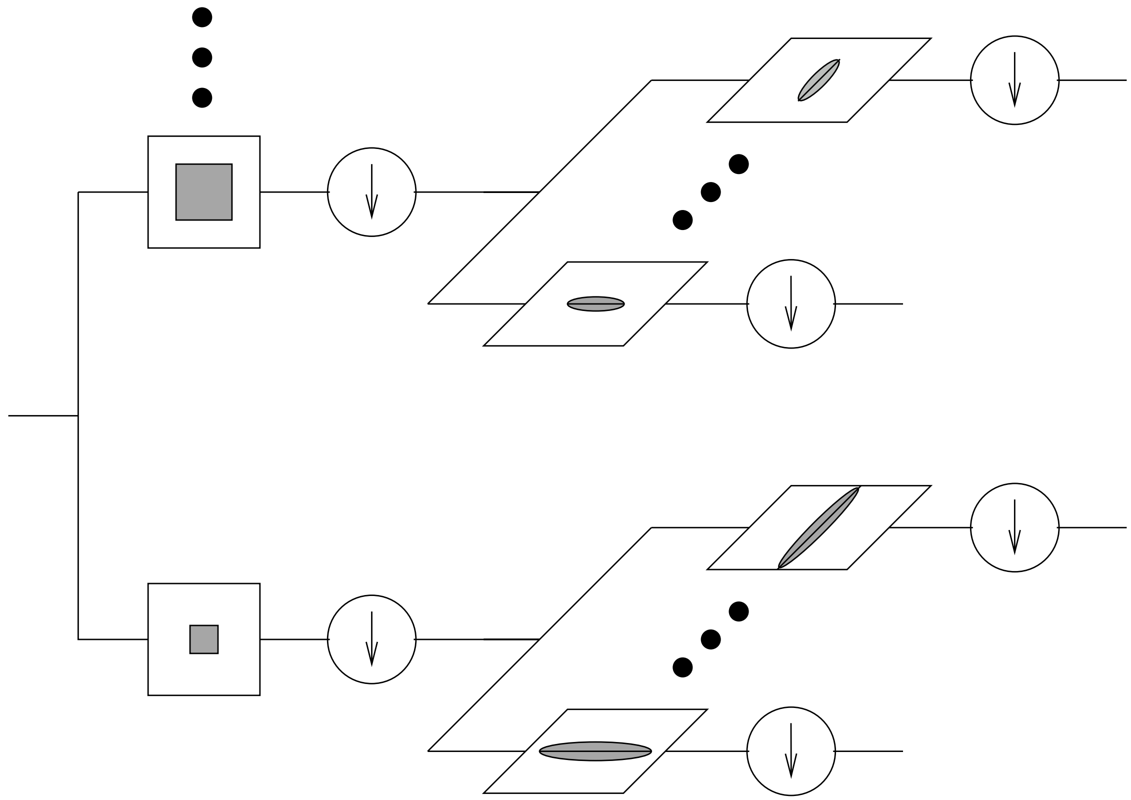
In a nutshell: contourlet transform is an efficient directional multiresolution expansion that is digital friendly!

contourlets = multiscale, local and directional contour segments

- Starts with a discrete-domain construction that is amenable to efficient algorithms, and then investigates its convergence to a continuous-domain expansion.

- The expansion is defined on rectangular grids $\Rightarrow$ seamless translation between the continuous and discrete worlds.

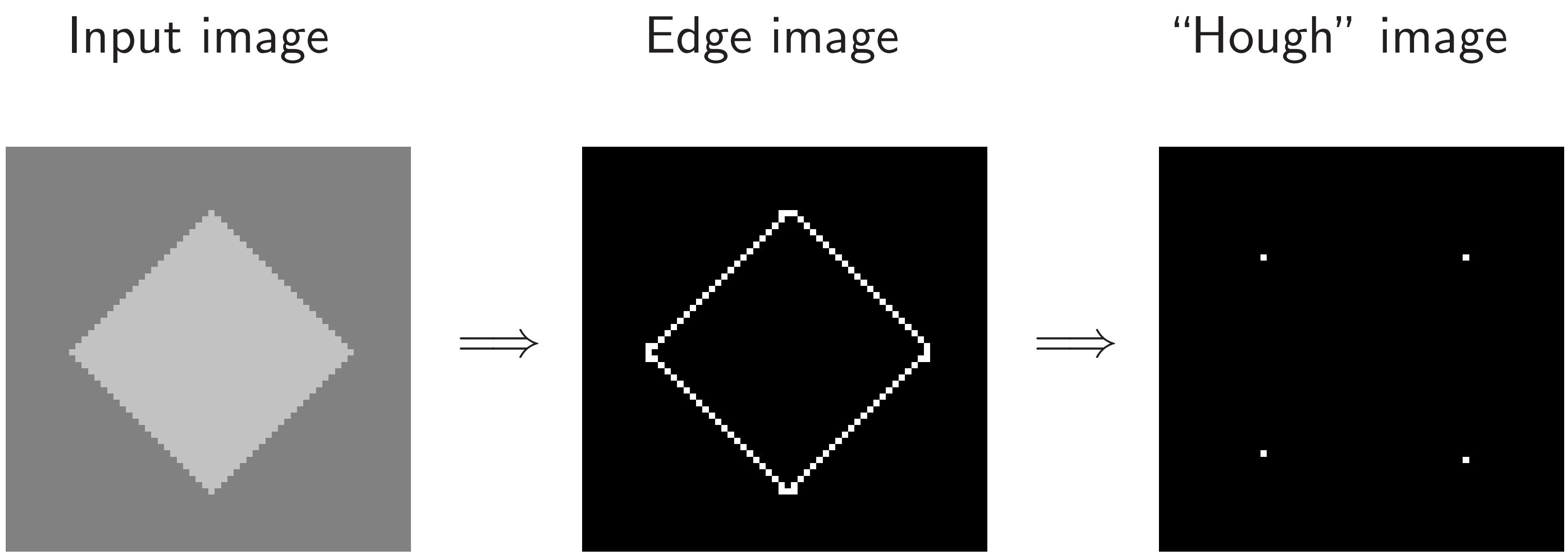
Discrete-Domain Construction using Filter Banks



Idea: Multiscale and Directional Decomposition

- Multiscale step: capture point discontinuities, followed by...
- Directional step: link point discontinuities into linear structures.

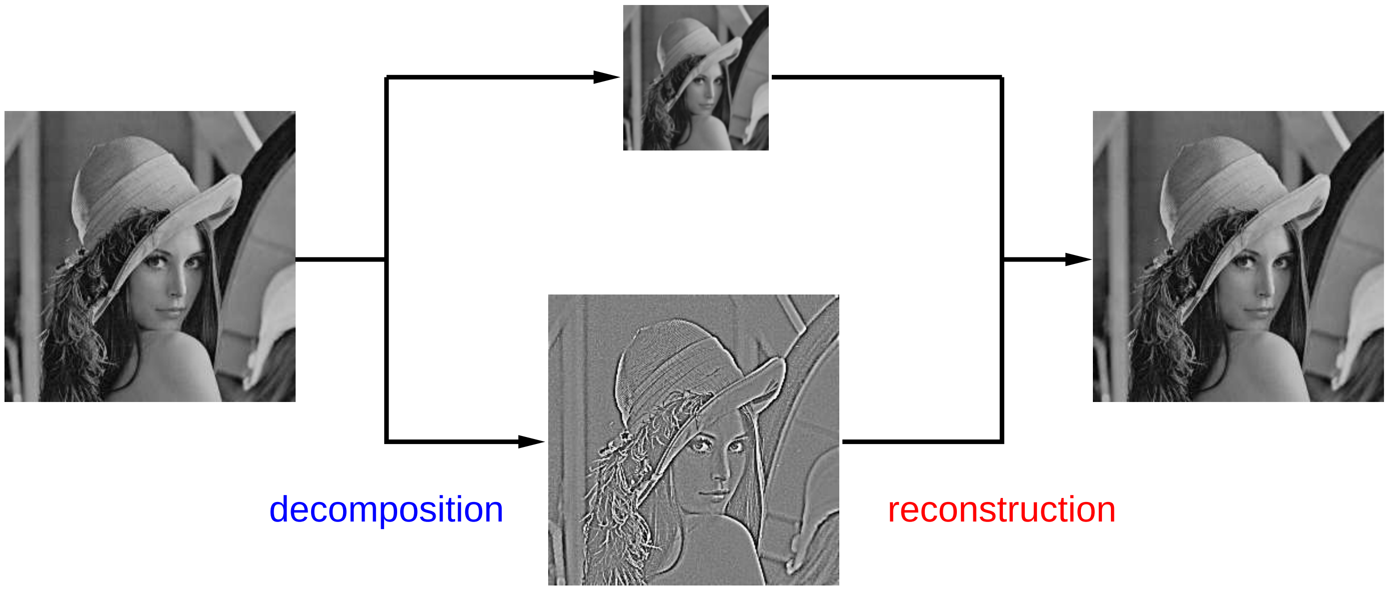
Analogy: Hough Transform in Computer Vision



Challenges:

- Perfect reconstruction.
- Fixed transform with low redundancy.
- Sparse representation for images with smooth contours.

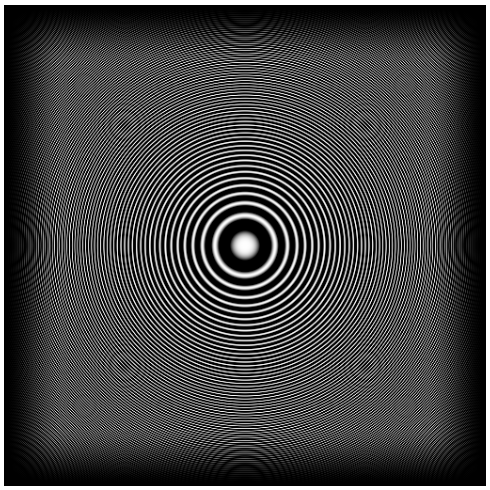
Multiscale Decomposition using Laplacian Pyramids

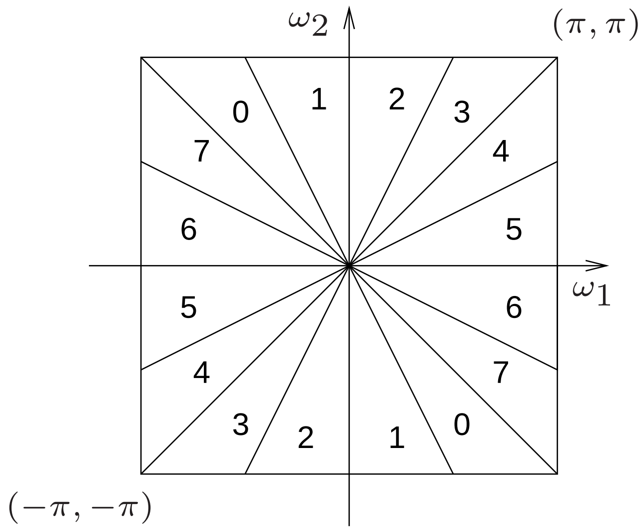


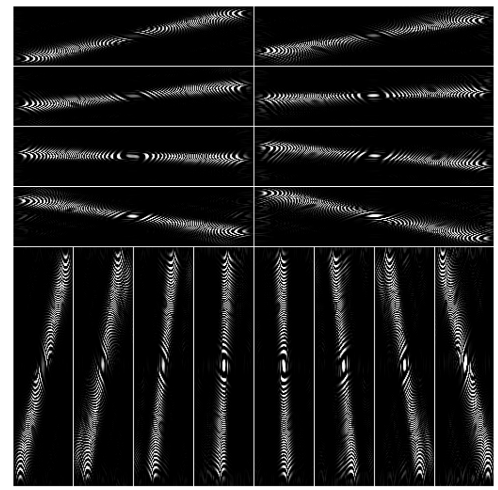
- **Reason:** avoid "frequency scrambling" due to $(\downarrow)$ of the HP channel.
- Laplacian pyramid as a frame operator $\rightarrow$ tight frame exists.
- New reconstruction: efficient filter bank for *dual* frame (pseudo-inverse).

Directional Filter Banks (DFB)

- **Feature:** division of 2-D spectrum into fine slices using tree-structured filter banks.



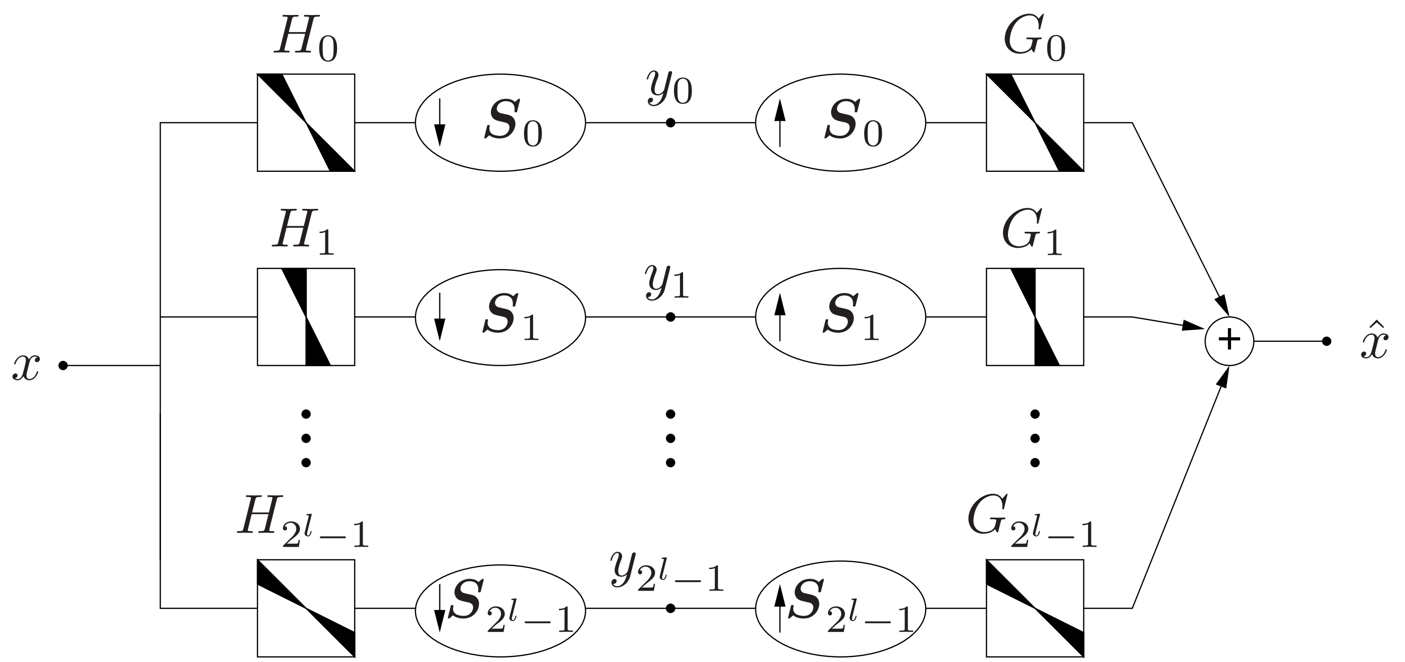




- **Background:** Bamberger and Smith ('92) cleverly used quincunx FB's, modulation and shearing.

- **We propose:**
  - a simplified DFB with fan FB's and shearing
  - use DFB to construct directional bases

Multichannel View of the Directional Filter Bank



Use two *separable* sampling matrices:

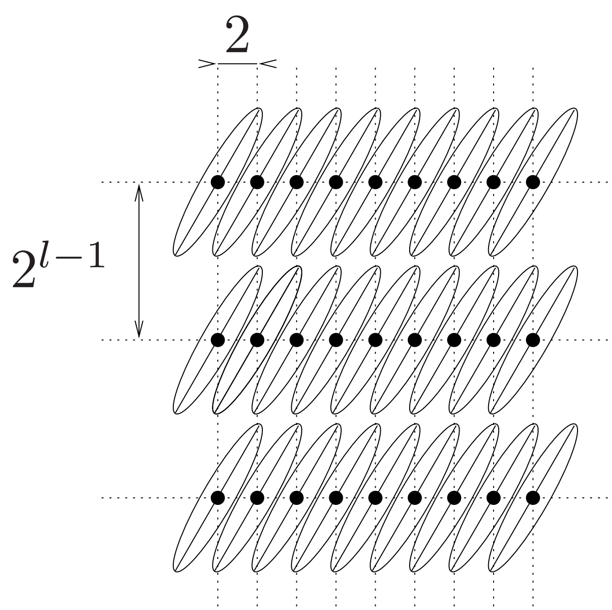
$$\boldsymbol{S}_k = \begin{cases} \begin{bmatrix} 2^{l-1} & 0 \\ 0 & 2 \end{bmatrix} & 0 \le k < 2^{l-1} \quad \text{(“near horizontal” direction)} \\ \begin{bmatrix} 2 & 0 \\ 0 & 2^{l-1} \end{bmatrix} & 2^{l-1} \le k < 2^l \quad \text{(“near vertical” direction)} \end{cases}$$

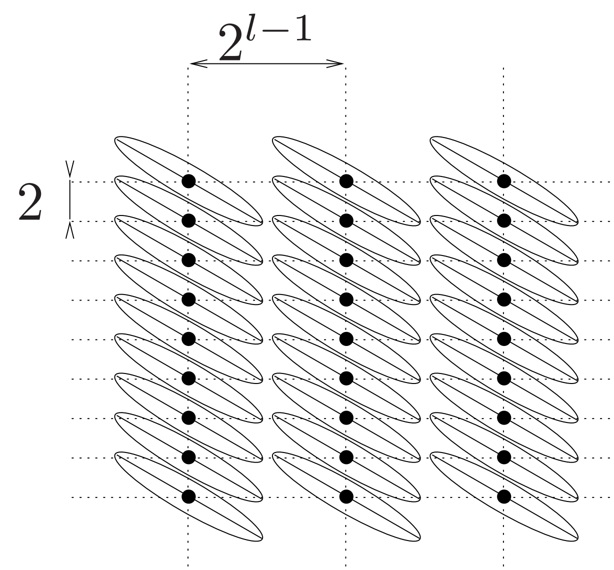
General Bases from the DFB

An l-levels DFB creates a local directional basis of $l^2(\mathbb{Z}^2)$:

$$\left\{ g_k^{(l)}[\cdot - S_k^{(l)} n] \right\}_{0 \le k < 2^l,\ n \in \mathbb{Z}^2}$$

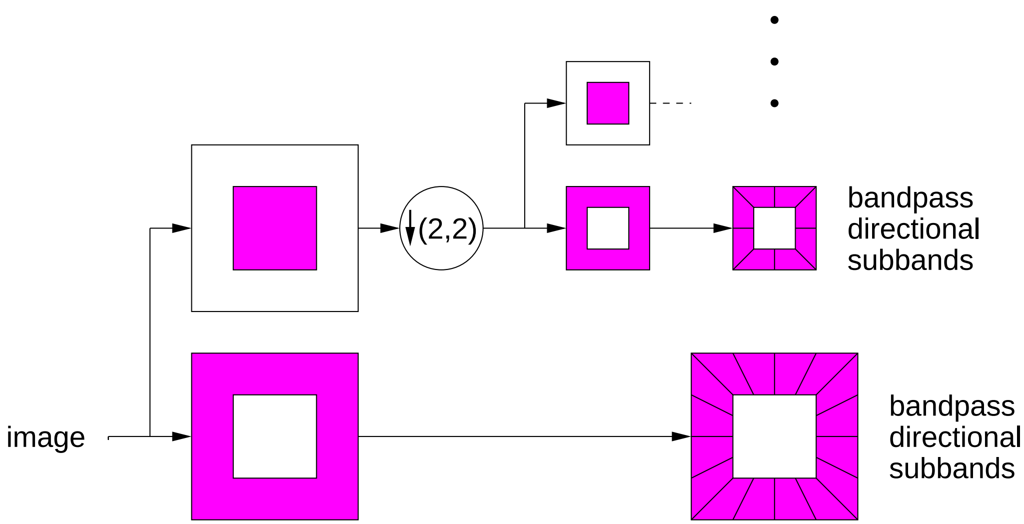
- $G_k^{(l)}$ are directional filters:
- Sampling lattices (spatial tiling):

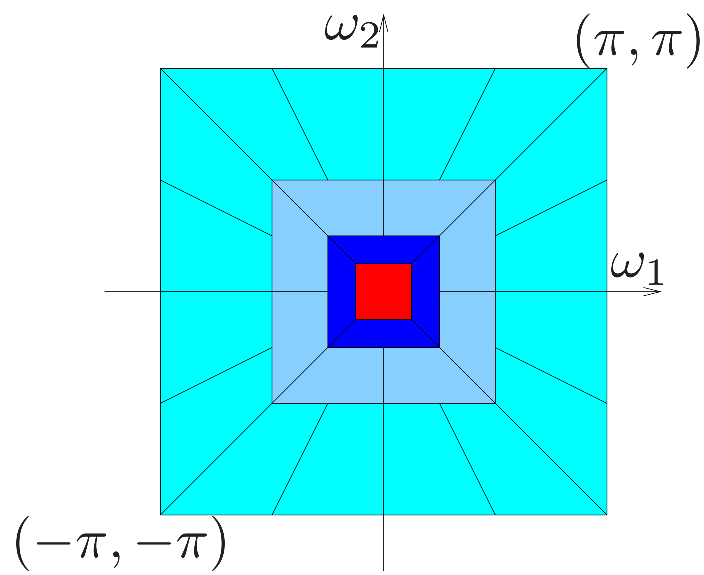




Pyramidal Directional Filter Banks (PDFB)

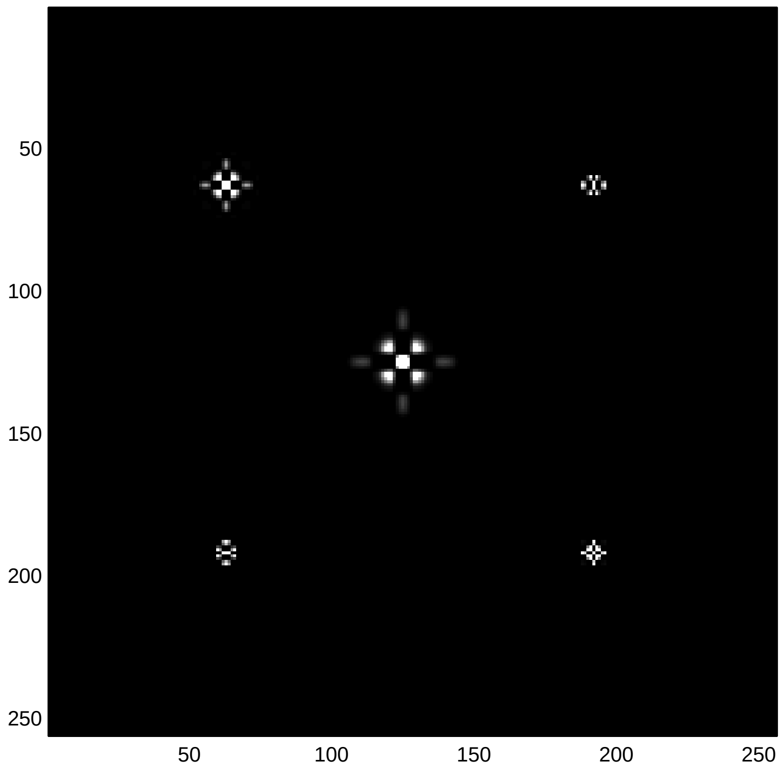
Motivation: + add multiscale into the directional filter bank
+ improve its non-linear approximation power.

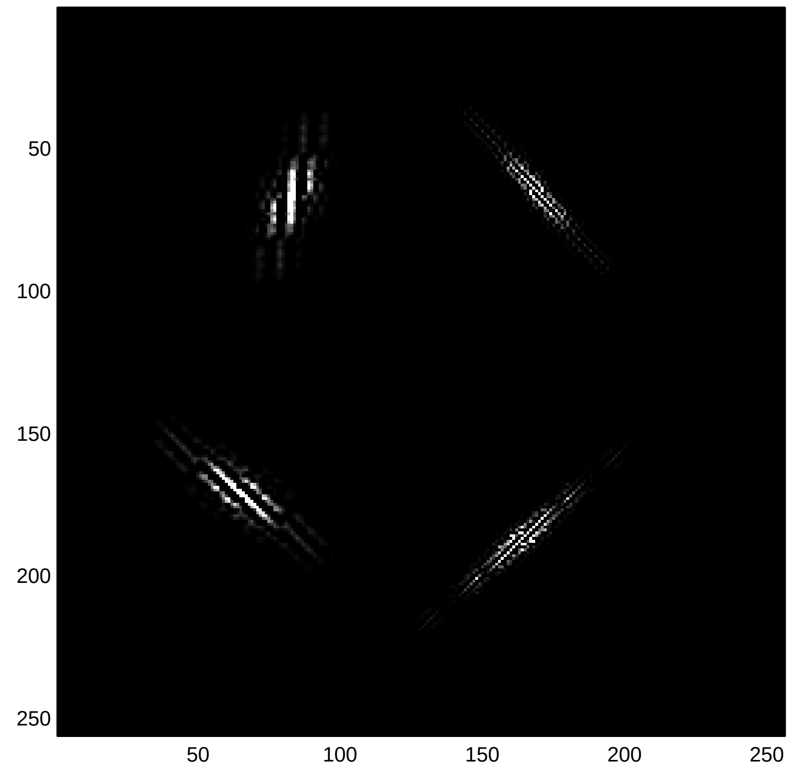




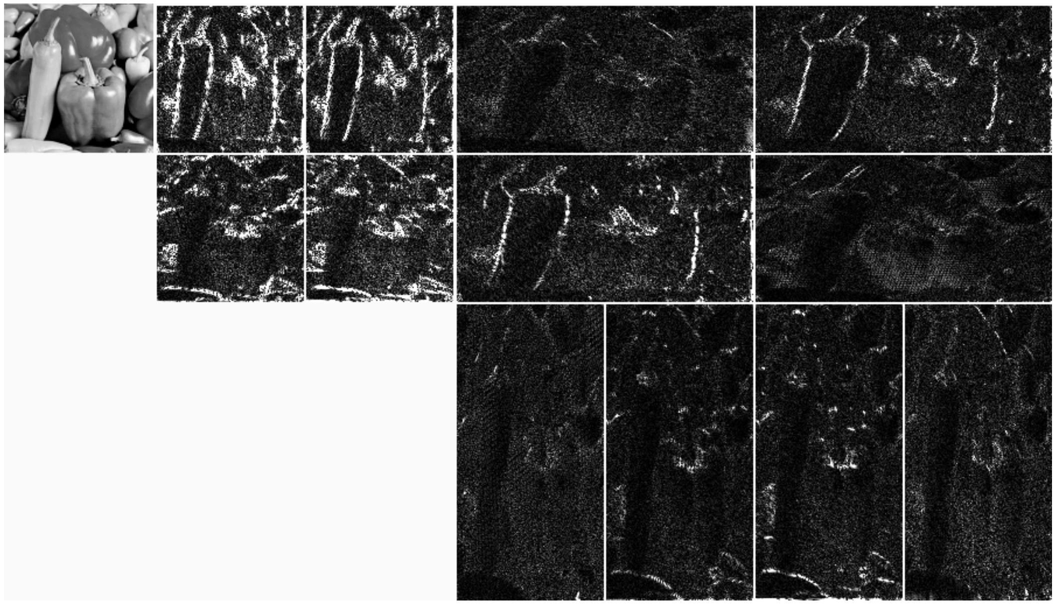
Properties: + Flexible multiscale and directional representation for images (can have different number of directions at each scale!)
+ Tight frame with small redundancy $(< 33\%)$
+ Computational complexity: $O(N)$ for N pixels.

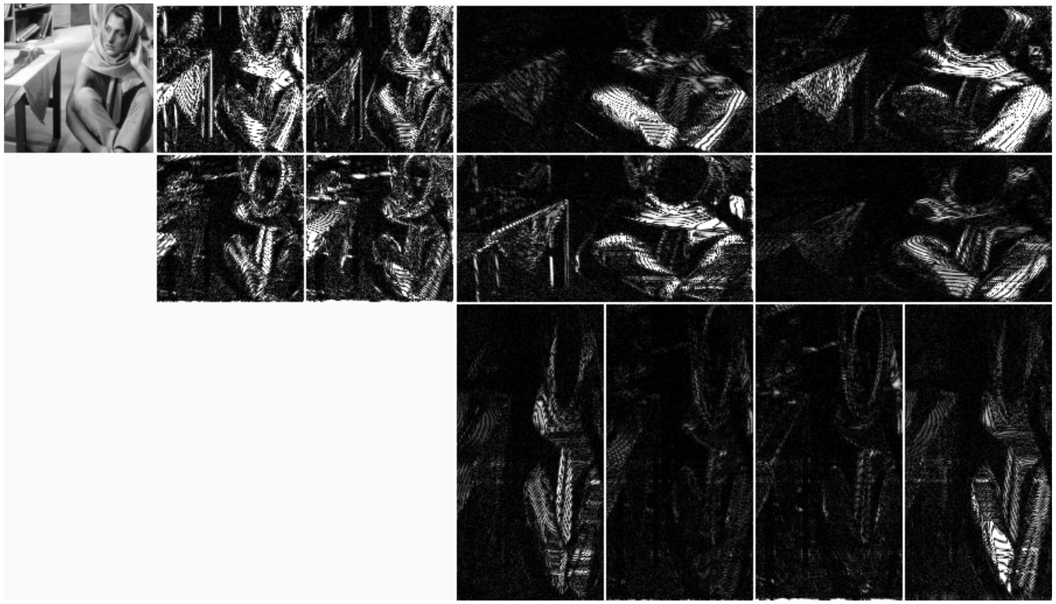
Wavelets vs. Contourlets





Examples of Discrete Contourlet Transform





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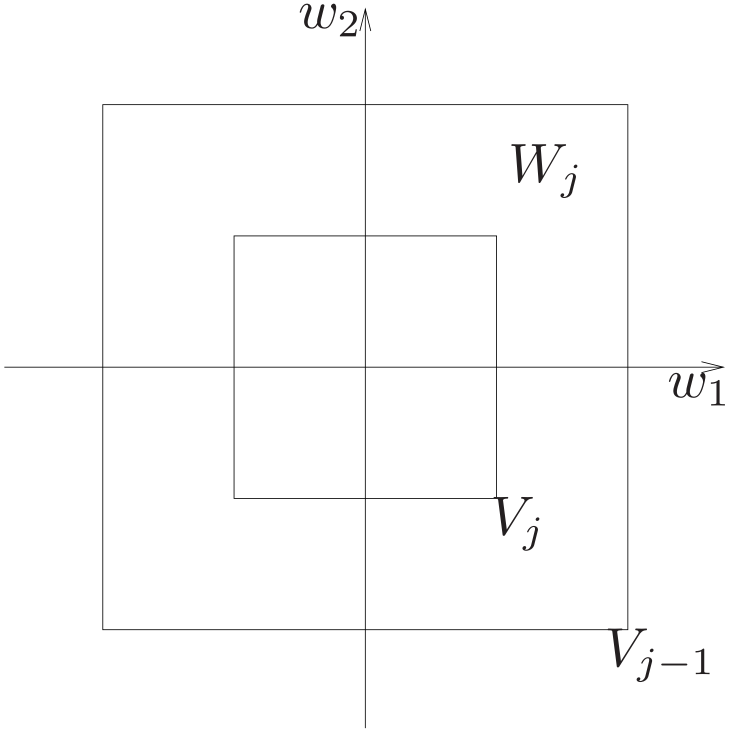
Multiresolution Analysis: Laplacian Pyramid



$$V_{j-1} = V_j \oplus W_j,$$

$$L^2(\mathbb{R}^2) = \bigoplus_{j\in\mathbb{Z}} W_j.$$

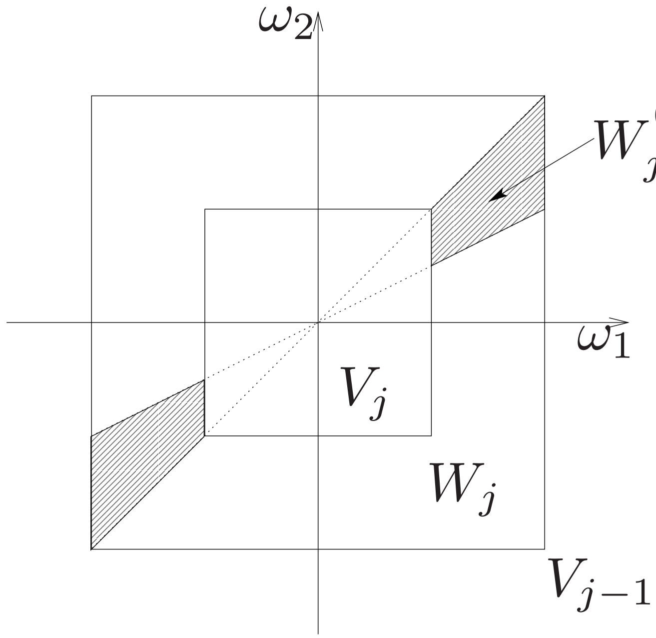
V_j has an orthogonal basis $\{\phi_{j,\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}^2}$, where

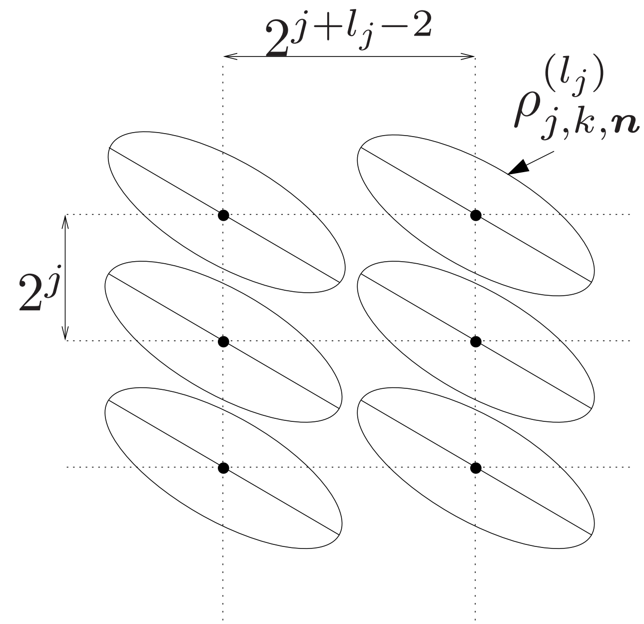
$$\phi_{j,\boldsymbol{n}}(t) = 2^{-j}\phi(2^{-j}\boldsymbol{t} - \boldsymbol{n}).$$

W_j has a tight frame $\{\mu_{j-1,\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}^2}$ where

$$\mu_{j-1,2\boldsymbol{n}+\boldsymbol{k}_i} = \psi^{(i)}_{j,\boldsymbol{n}}, \quad i = 0,\ldots,3.$$

Directional Multiresolution Analysis: LP + DFB





$$W_j = \bigoplus_{k=0}^{2^{l_j}-1} W_{j,k}^{(l_j)}$$

$W_{j,k}^{(l_j)}$ has a tight frame $\left\{\rho_{j,k,\boldsymbol{n}}^{(l)}\right\}_{\boldsymbol{n}\in\mathbb{Z}^2}$ where

$$\rho_{j,k,\boldsymbol{n}}^{(l)}(\boldsymbol{t}) = \sum_{\boldsymbol{m}\in\mathbb{Z}^2} \underbrace{g_k^{(l)}[\boldsymbol{m} - \boldsymbol{S}_k^{(l)}\boldsymbol{n}]}_{\text{DFB basis}} \underbrace{\mu_{j-1,\boldsymbol{m}}(\boldsymbol{t})}_{\text{LP frame}} = \rho_{j,k}^{(l)}(\boldsymbol{t} - 2^{j-1}\boldsymbol{S}_k^{(l)}\boldsymbol{n}).$$

Contourlet Frames

Theorem (Contourlet Frames) [DoV:03].
$\left\{\rho^{(l_j)}_{j,k,\boldsymbol{n}}\right\}_{j\in\mathbb{Z},\ 0\le k<2^{l_j},\ \boldsymbol{n}\in\mathbb{Z}^2}$ is a tight frame of $L^2(\mathbb{R}^2)$ for finite l_j.

Theorem (Connection with Filter Banks) [DoV:04]
Suppose $x[\boldsymbol{n}] = \langle f, \phi_{L,\boldsymbol{n}}\rangle$, $\boldsymbol{n} \in \mathbb{Z}^2$, for some function $f \in L^2(\mathbb{R}^2)$. Furthermore, suppose

$$x \stackrel{\text{PDFB}}{\longrightarrow} (a_J,\ d^{(l_j)}_{j,k})_{j=1,\dots,J;\ k=0,\dots,2^{l_j}-1}$$

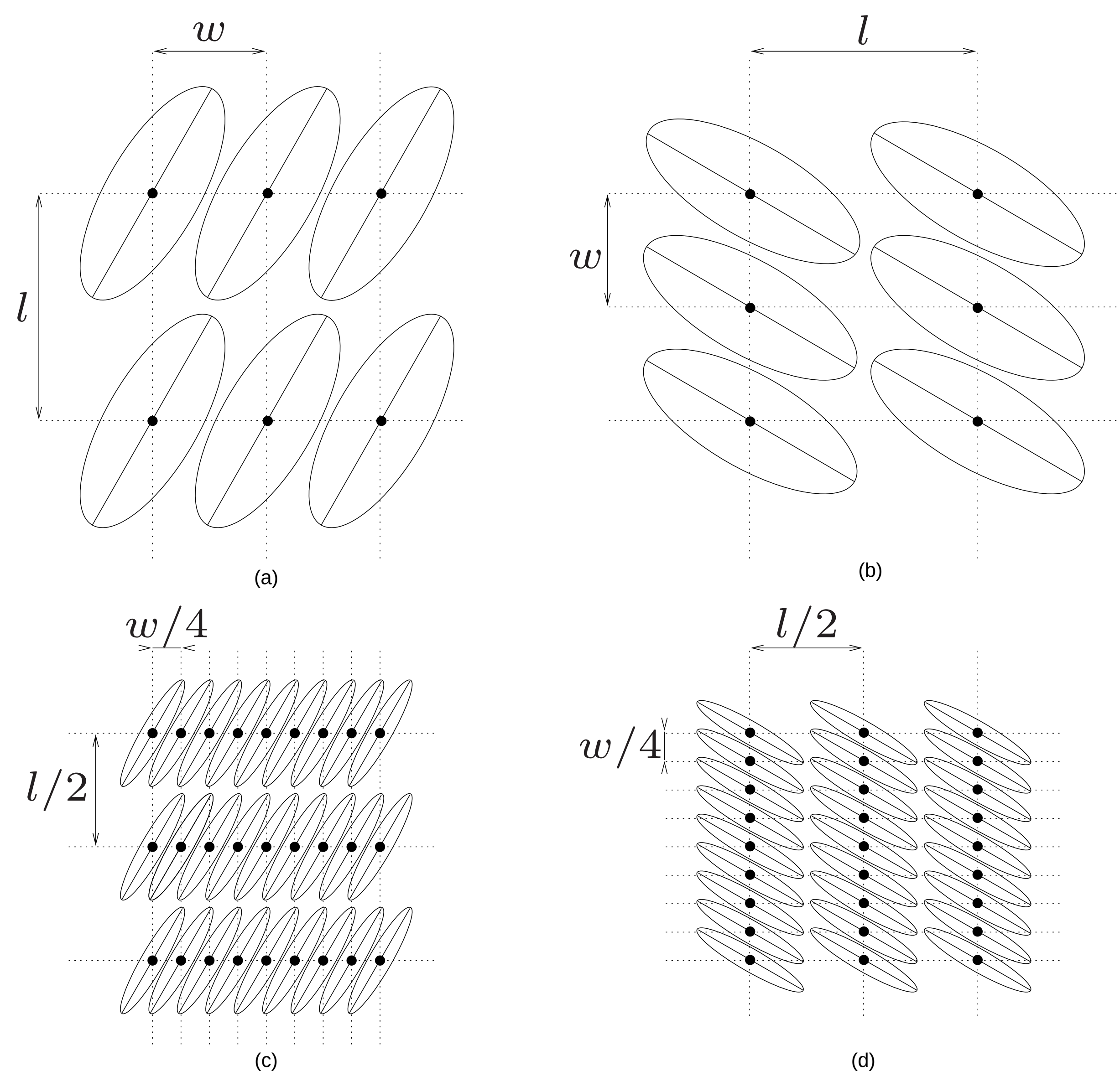
where a_J is the lowpass subband, and $d^{(l_j)}_{j,k}$ are bandpass directional subbands. Then

$$a_J[\boldsymbol{n}] = \langle f, \phi_{L+J,\boldsymbol{n}}\rangle$$

$$d^{(l_j)}_{j,k}[\boldsymbol{n}] = \langle f, \rho^{(l_j)}_{L+j,k,\boldsymbol{n}}\rangle$$

for $j = 1,\dots,J;\ k = 0,\dots,2^{l_j}-1,\ \boldsymbol{n} \in \mathbb{Z}^2$.

Sampling Grids of Contourlets

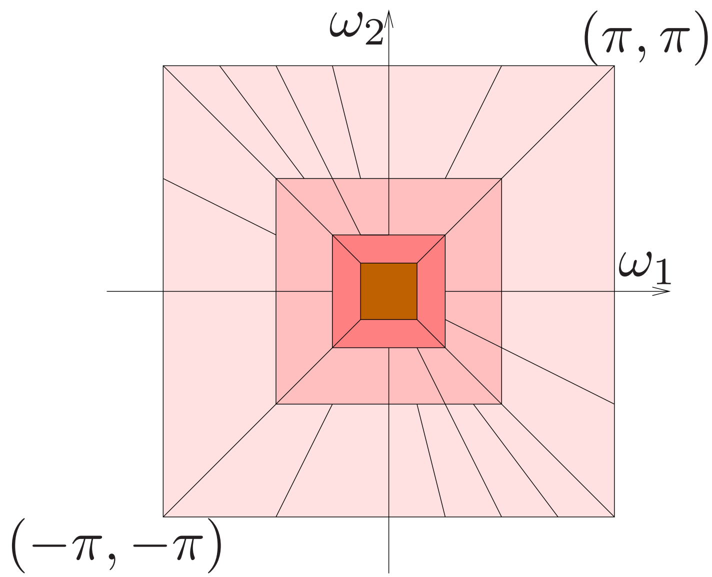


Contourlet Features

- Defined via iterated filter banks $\Rightarrow$ fast algorithms, tree structures, etc.
- Defined on rectangular grids $\Rightarrow$ seamless translation between continuous and discrete worlds.
- Different contourlet kernel functions $(\rho_{j,k})$ for different directions.
- These functions are defined iteratively via filter banks.
- With FIR filters $\Rightarrow$ compactly supported contourlet functions.

Contourlet Packets

- Adaptive scheme to select the "best" tree for directional decomposition.



- Contourlet packets $\Rightarrow$ directional multiresolution elements with different shapes (aspect ratios).
  - They do not necessary satisfy the parabolic relation.
  - They can include wavelets!

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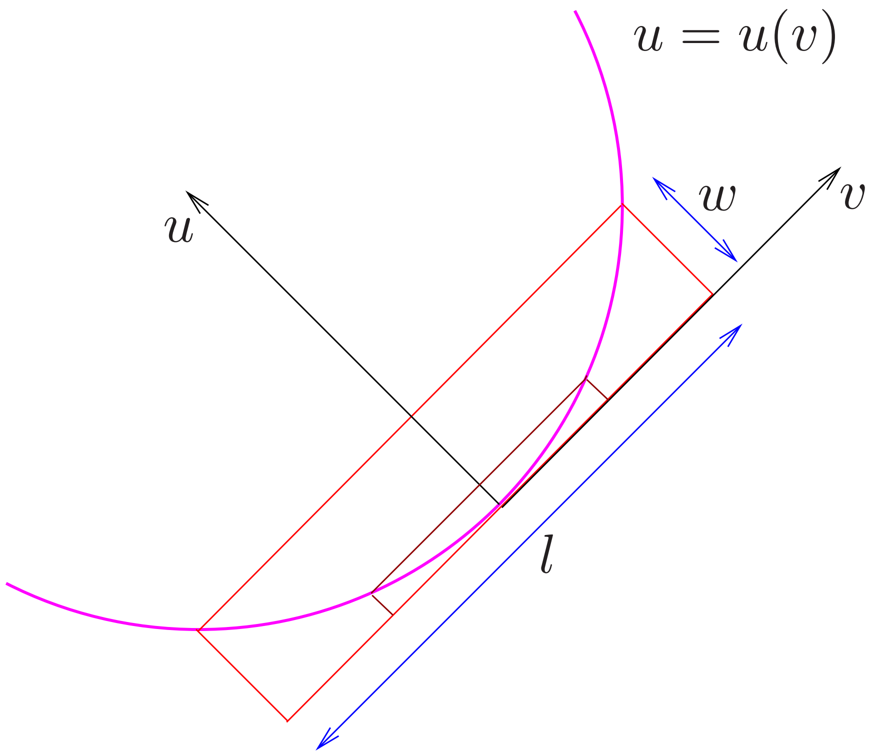
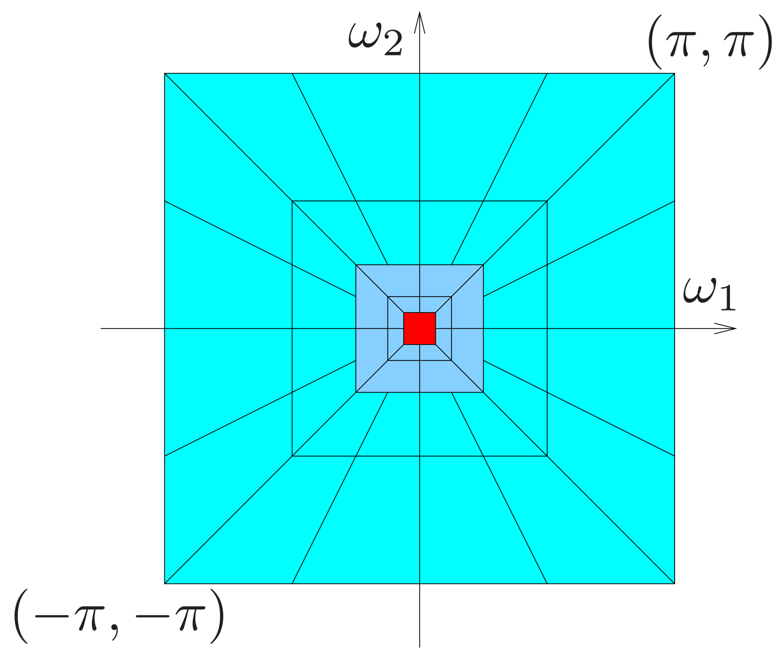
Contourlets with Parabolic Scaling

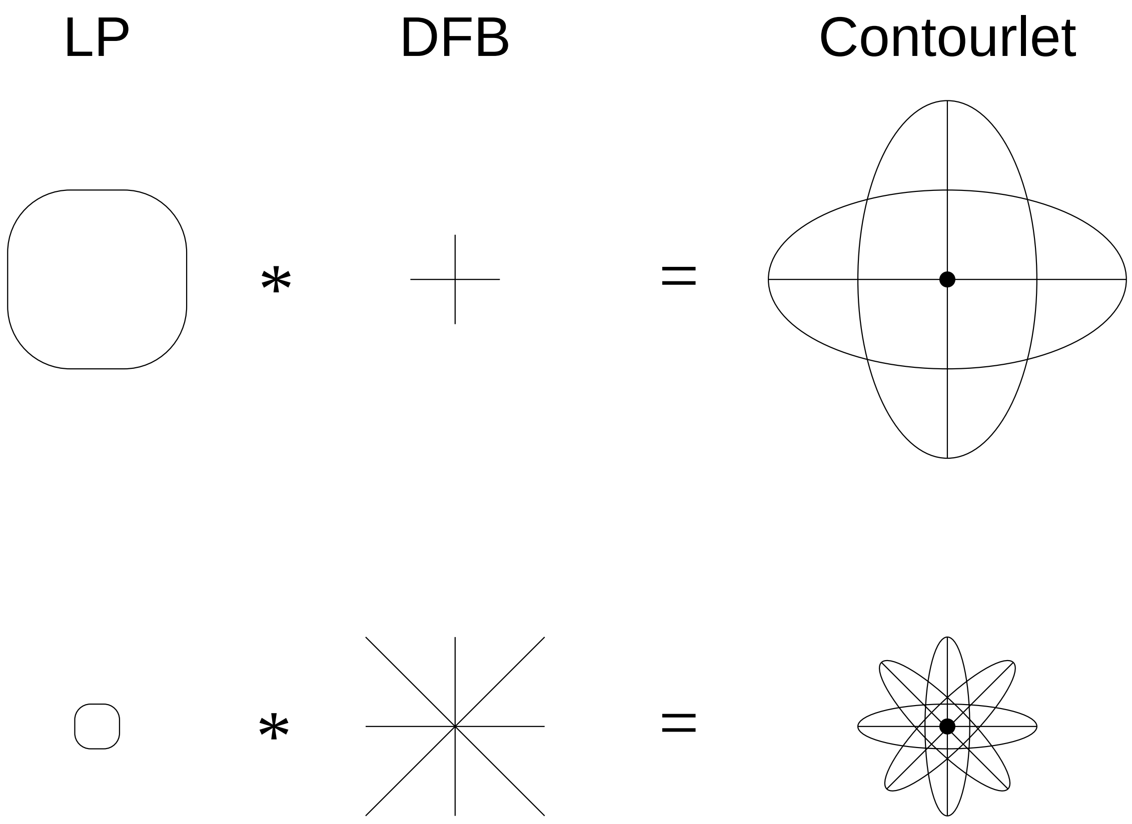
Support size of the contourlet function $\rho_{j,k}^{l_j}$: $width \approx 2^j$ and $length \approx 2^{l_j + j}$

To satisfy the parabolic scaling (for C^2 curved singularities):
$width \propto length^2$, simply set:
the number of directions in the PDFB is doubled at every ***other*** *finer scale*.



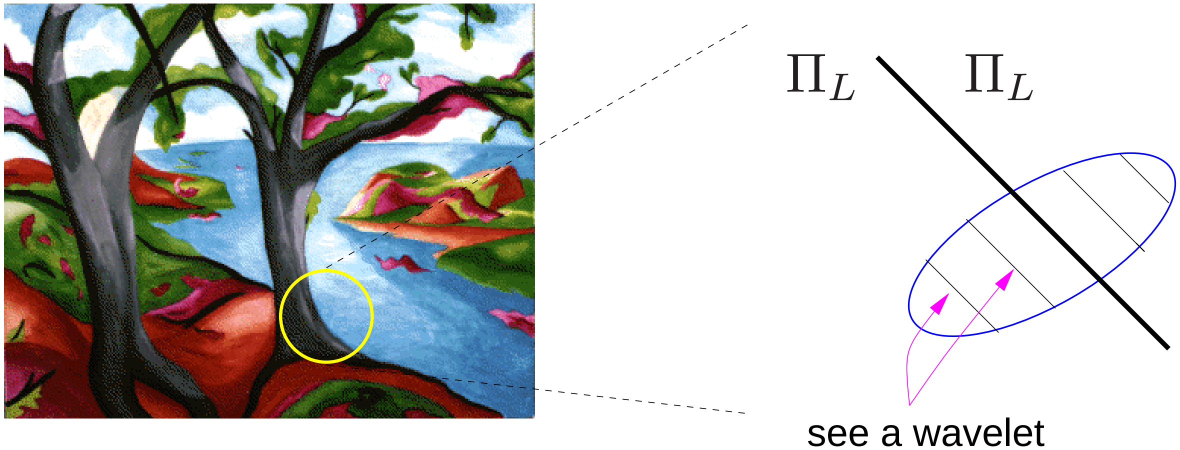


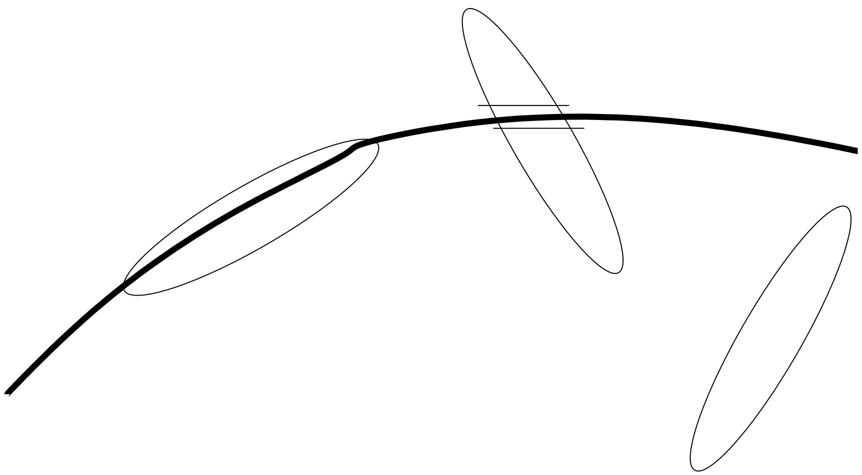
Supports of Contourlet Functions



Key point: Each generation doubles spatial resolution
as well as angular resolution.

Contourlet Approximation





Desire: Fast decay as contourlets turn away from the discontinuity direction

Key: Directional vanishing moments (DVMs)

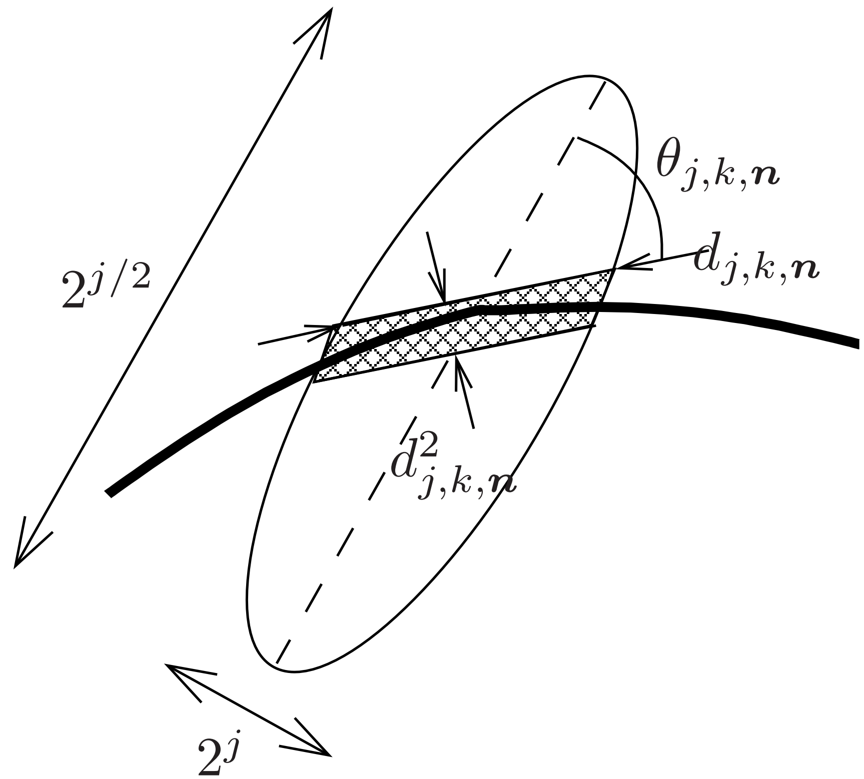
Geometrical Intuition

At scale 2^j $(j \ll 0)$:

$$width \approx 2^j$$

$$length \approx 2^{j/2}$$

$$\#directions \approx 2^{-j/2}$$

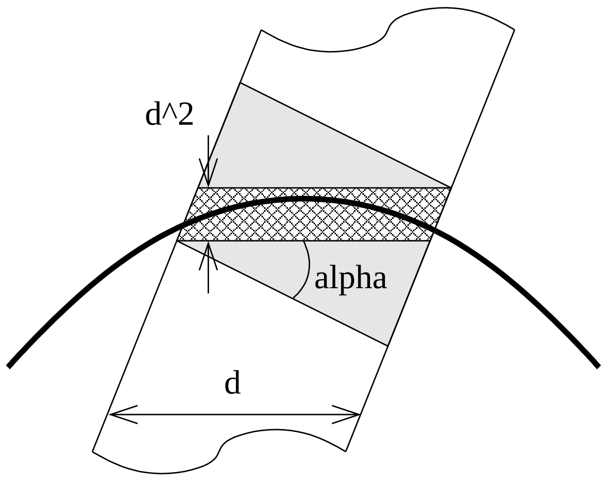


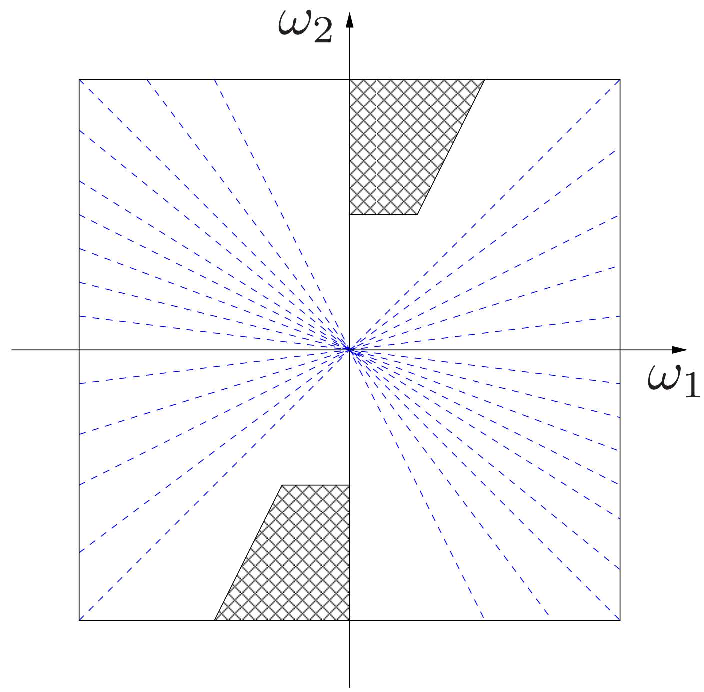
$$|\langle f, \rho_{j,k,\boldsymbol{n}} \rangle| \sim 2^{-3j/4} \cdot d^3_{j,k,\boldsymbol{n}}$$

$$d_{j,k,\boldsymbol{n}} \sim 2^j / \sin\theta_{j,k,\boldsymbol{n}} \sim 2^{j/2}\tilde{k}^{-1} \quad \text{for} \quad \tilde{k} = 1, \ldots, 2^{-j/2}$$

$$\Longrightarrow |\langle f, \rho_{j,\tilde{k},\boldsymbol{n}} \rangle| \sim 2^{3j/4}\tilde{k}^{-3}$$

How Many DVMs Are Sufficient?





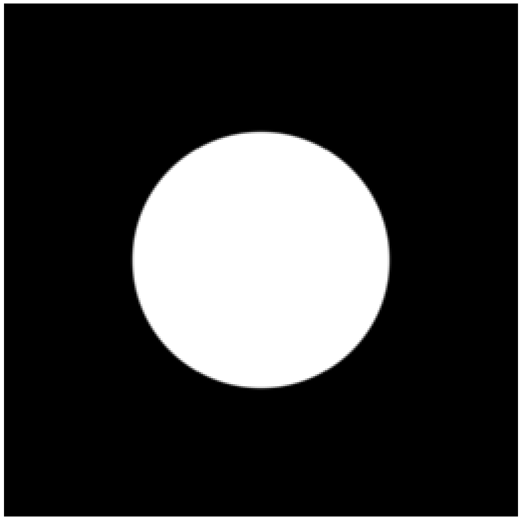
Sufficient if the gap to a direction with DVM:

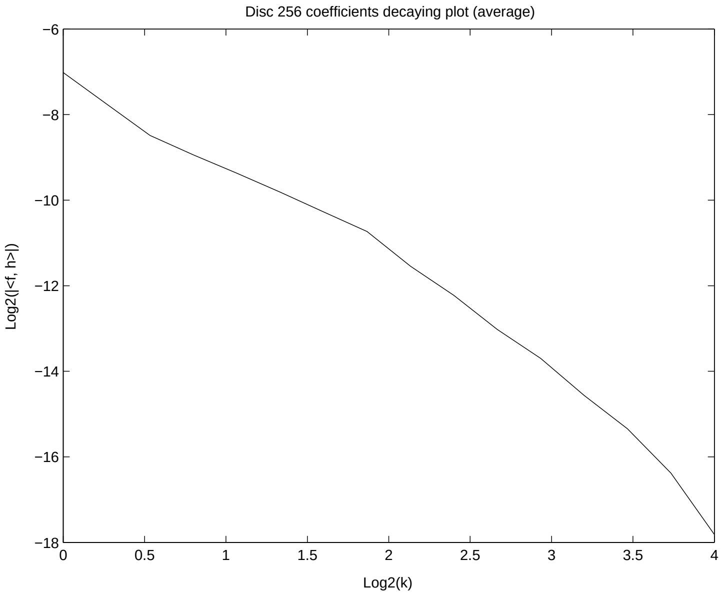
$$\alpha \lesssim d \sim 2^{j/2}\tilde{k}^{-1} \quad \text{for} \quad \tilde{k} = 1, \ldots, 2^{-j/2}$$

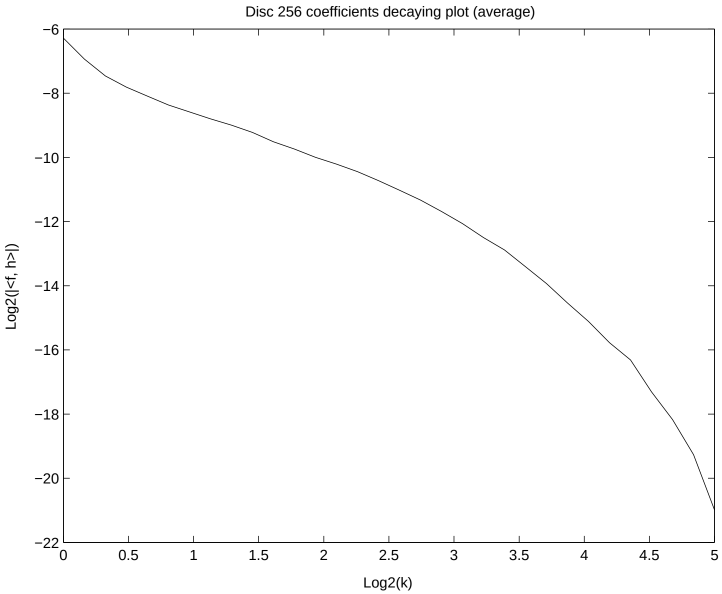
This condition can be replaced with fast decay in frequency across directions.

It is still an open question if there is an FIR filter bank that satisfies the sufficient DVM condition

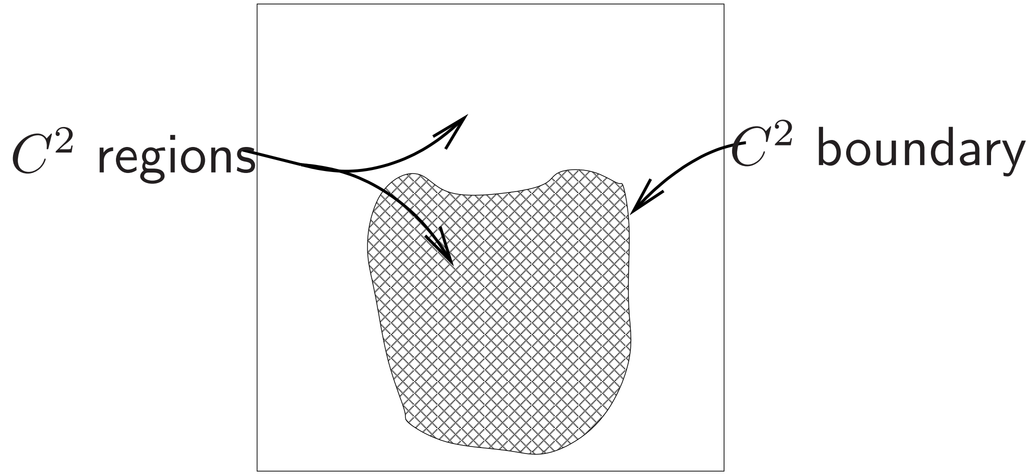
Experiments with Decay Across Directions using Near Ideal Frequency Filters







Nonlinear Approximation Rates



Under the (ideal) sufficient DVM condition

$$|\langle f, \rho_{j,\tilde{k},\boldsymbol{n}} \rangle| \sim 2^{3j/4} \tilde{k}^{-3}$$

with number of coefficients $N_{j,\tilde{k}} \sim 2^{-j/2}\tilde{k}$. Then

$$\|f - \hat{f}_M^{(contourlet)}\|^2 \sim (\log M)^3 M^{-2}$$

While $\|f - \hat{f}_M^{(Fourier)}\|^2 \sim O(M^{-1/2})$ and $\|f - \hat{f}_M^{(wavelet)}\|^2 \sim O(M^{-1})$

Non-linear Approximation Experiments

Image size $= 512 \times 512$. Keep $M = 4096$ coefficients.

Original image

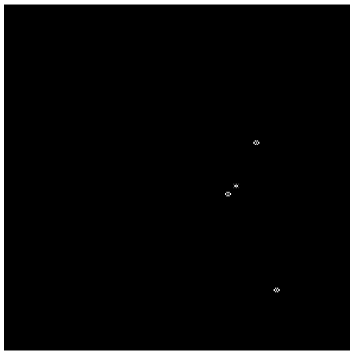
Wavelets:
PSNR = 24.34 dB

Contourlets:
PSNR = 25.70 dB

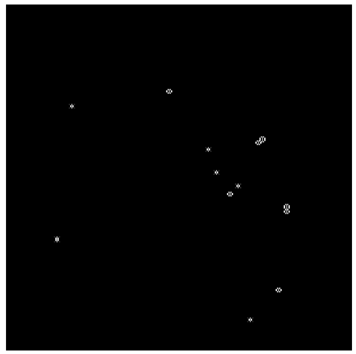
Detailed Non-linear Approximations

Wavelets

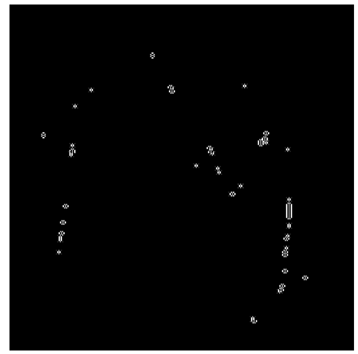
M = 4



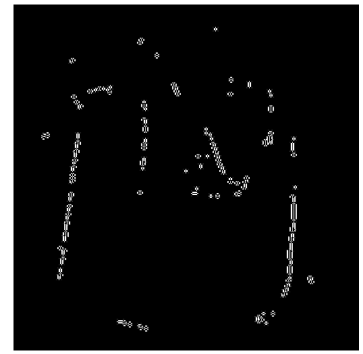
M = 16



M = 64

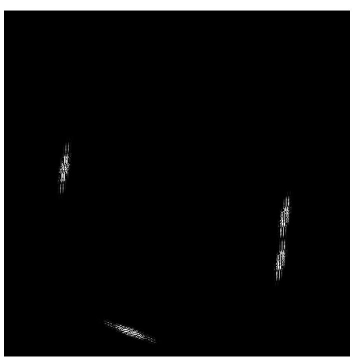


M = 256

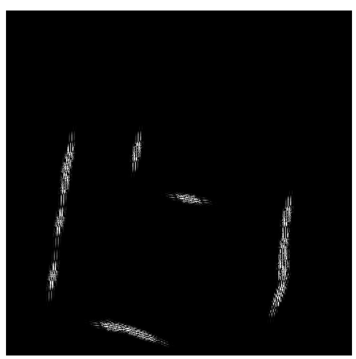


Contourlets

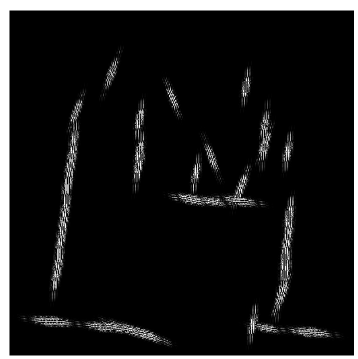
M = 4



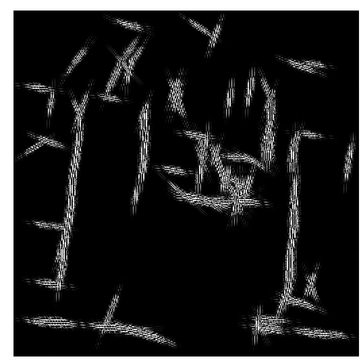
M = 16



M = 64

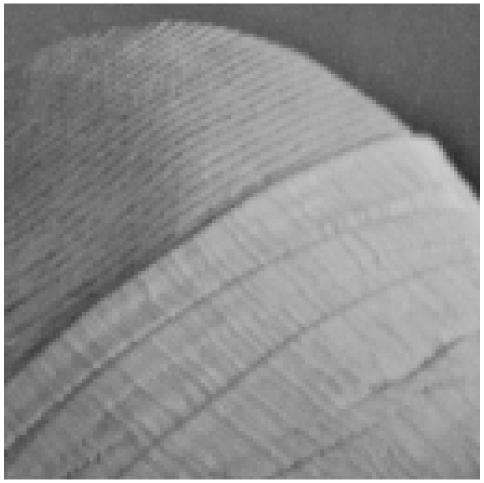


M = 256

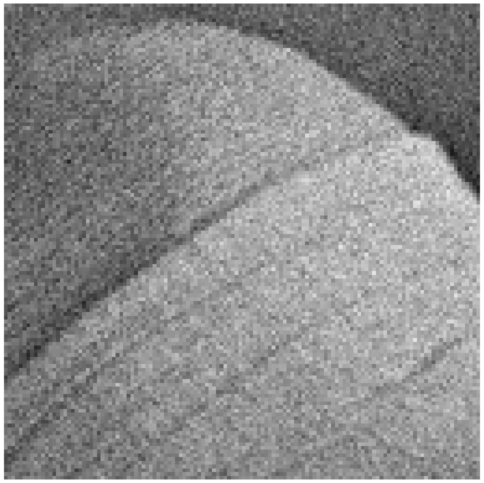


Denoising Experiments

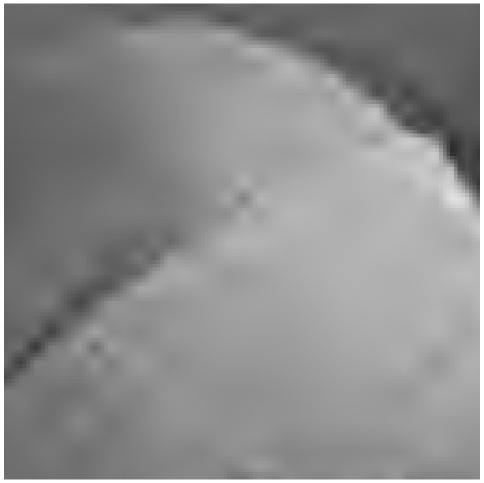
original image



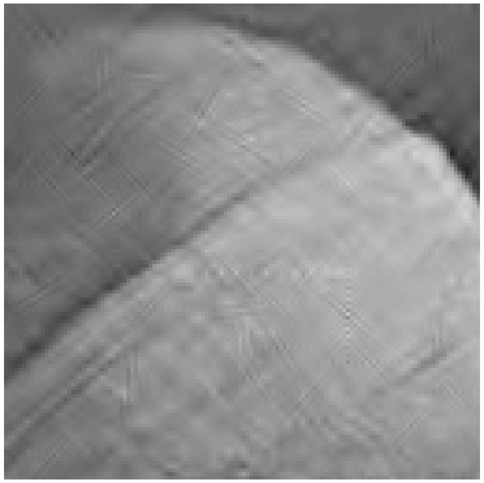
noisy image (SNR = 9.55 dB)



wavelet denoising (SNR = 13.82 dB)



contourlet denoising (SNR = 15.42 dB)



Embedded Structure for Compression and Modeling

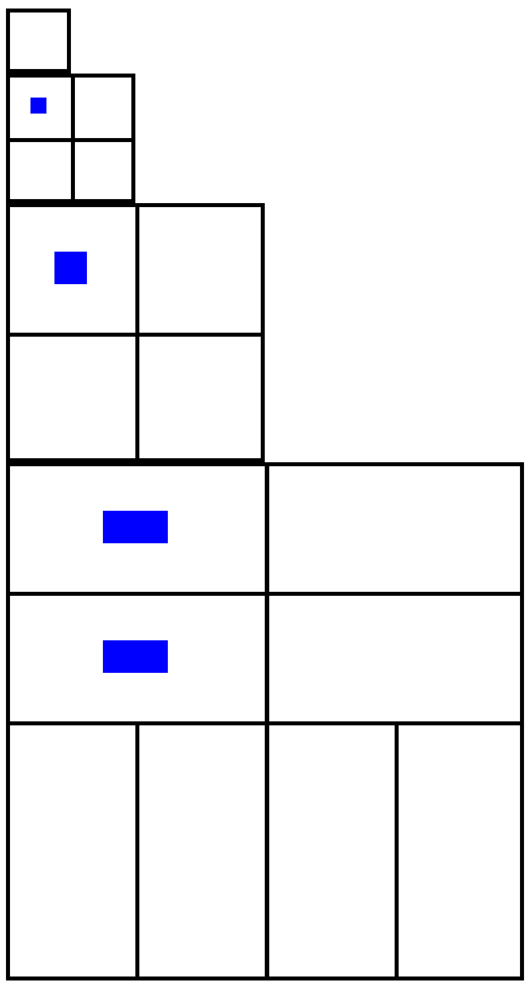
- So far, *best*-M term approximation:

  $$\hat{f}_M = \sum_{\lambda \in I_M} c_\lambda \rho_\lambda, \quad \text{where } I_M \text{ is the set of indexes of the } M\text{-largest } |c_\lambda|.$$

- For compression, additional cost required to specify I_M
  - **Naive approach:** $M \cdot \log_2 N$ bits
  - **With embedded tree (wavelets):** M bits

- Embedded trees for wavelets are crucial in state-of-the-art image compression (EZW,...), rate-distortion analysis (Cohen et al.), and multiscale statistical modeling (Baraniuk et al.)

Contourlet Embedded Tree Structure



Embedded tree data structure for contourlet coefficients:

successively locate the position and direction of image contours.

Since significant contourlet coefficients are organized in trees, best M-tree approximation (using M-node tree):

$$\|f - \hat{f}^{(contourlet)}_{M-tree}\|^2 \approx (\log M)^3 M^{-2}$$

$$\Rightarrow D(R) \approx (\log R)^3 R^{-2}$$

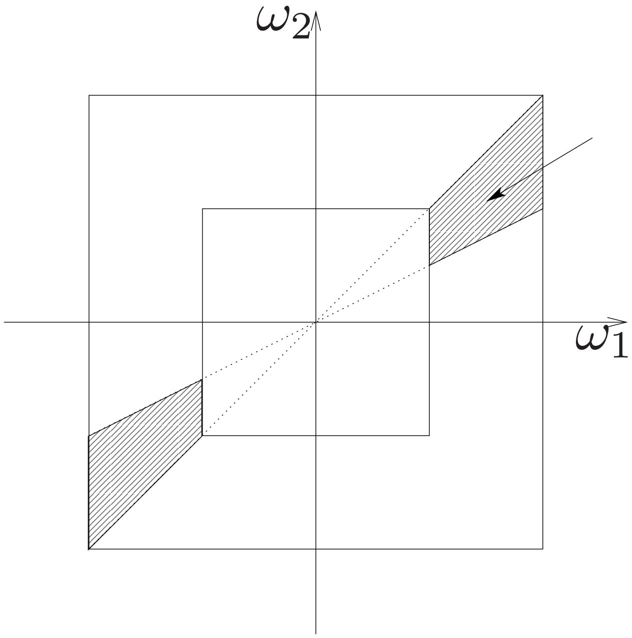
Outline

1. Motivation
2. Discrete-domain construction using filter banks
3. Contourlets and directional multiresolution analysis
4. Contourlet approximation
5. Contourlet filter design with directional vanishing moments

Filter Bank Design Problem

$$\rho_{j,k}^{(l)}(\boldsymbol{t}) = \sum_{\boldsymbol{m} \in \mathbb{Z}^2} c_k^{(l)}[\boldsymbol{m}] \phi_{j-1,\boldsymbol{m}}(\boldsymbol{t})$$

$\rho_{j,k}^{(l)}(\boldsymbol{t})$ has an L-order DVM along direction (u_1, u_2)
$\Leftrightarrow C_k^{(l)}(z_1, z_2) = (1 - z_1^{u_2} z_2^{-u_1})^L R(z_1, z_2)$



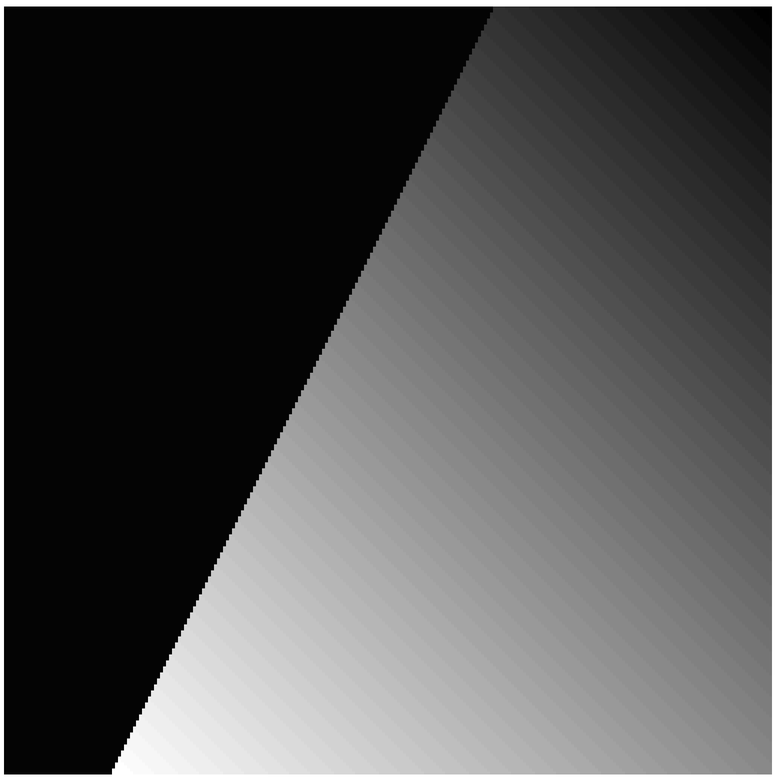
So far: Use good frequency selectivity to approximate DVMs.

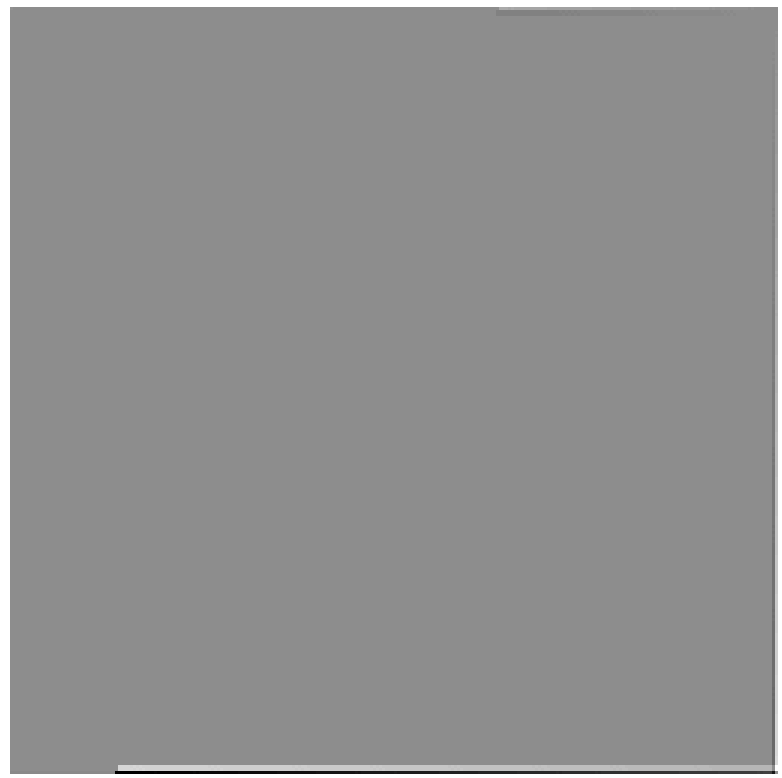
Draw back: long filters...

Next: Design short filters that lead to many DVMs as possible.

Filters with DVMs = Directional Annihilating Filters

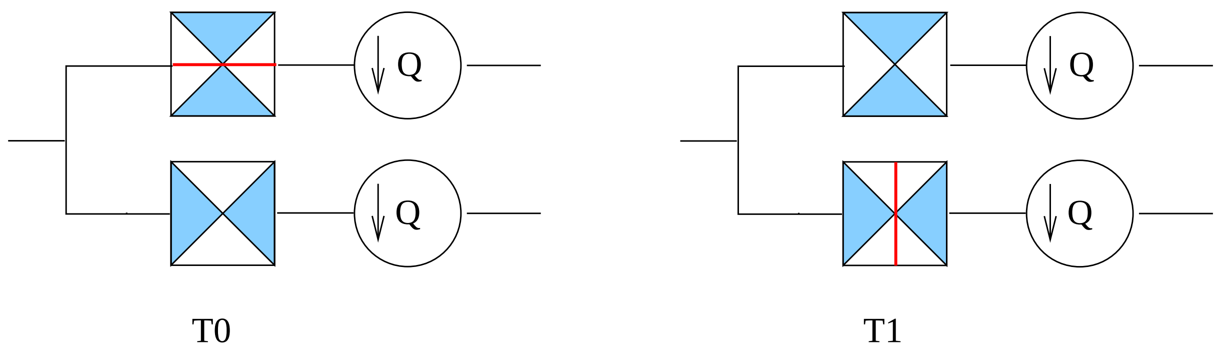
Input image and after being filtered by a directional annihilating filter





Perfect Reconstruction Two-Channel FBs with DVMs

Filter bank with order-L horizontal or vertical DVM:

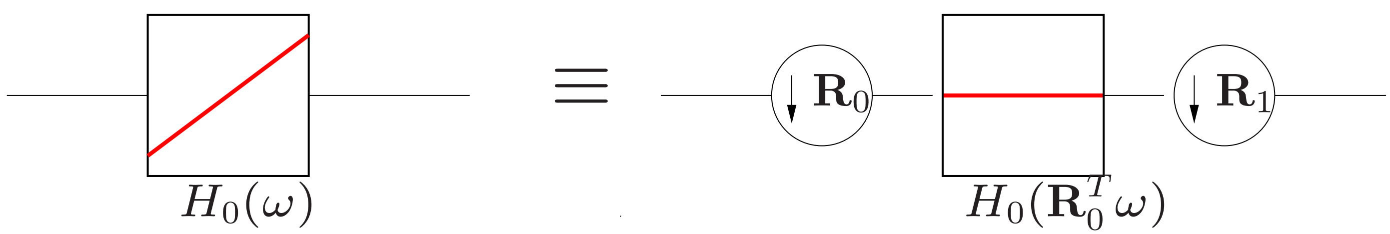


Filter design amounts to solve

$$P(\boldsymbol{z}) + P(-\boldsymbol{z}) = 2,$$

where $P(\boldsymbol{z}) = H_0(\boldsymbol{z})G_0(\boldsymbol{z}) = (1 - z_1)^L R(\boldsymbol{z})$.

To get DVMs at other directions: Shearing or change of variables



Complete Characterization

Proposition [Cunha-D., 04]. Any FIR solution of

$$(1 - z_1)^L R(\boldsymbol{z}) + (1 + z_1)^L R(-\boldsymbol{z}) = 2 \tag{1}$$

can be written as

$$R(\boldsymbol{z}) = A_L(z_1) + (1 + z_1)^L B(\boldsymbol{z})$$

where $A_L(z_1)$ is the minimum degree 1-D polynomial in z_1 that solves (1)

$$A_L(z_1) = \sum_{i=0}^{L-1} \binom{L + i - 1}{L - 1} 2^{-(L+i-1)} (1 - z_1)^i,$$

and $B(\boldsymbol{z})$ satisfy $B(\boldsymbol{z}) + B(-\boldsymbol{z}) = 0.$

Proof. Applying the Bezout theorem for each z_2.

Design via Mapping (Cunha-D., 2004)

To avoid 2D factorization...

1. Start with a 1-D solution:

$$P^{(1\text{D})}(z) + P^{(1\text{D})}(-z) = 2 \quad \text{and} \quad P^{(1\text{D})}(z) = H_0^{(1\text{D})}(z) G_0^{(1\text{D})}(z)$$

2. Apply a 1-D to 2-D mapping to each filter:

$$F^{(1\text{D})}(z) \to F^{(2\text{D})}(\boldsymbol{z}) = F^{(1\text{D})}(M(z_1, z_2))$$

- If $M(\boldsymbol{z}) = -M(-\boldsymbol{z})$ then PR is preserved

$$P^{(2\text{D})}(\boldsymbol{z}) + P^{(2\text{D})}(-\boldsymbol{z}) = 2$$

- Suppose $H^{(1\text{D})}(z) = (1+z)^k R_1(z)$ and $M(\boldsymbol{z}) + 1 = (1 - z_1)^l m(\boldsymbol{z})$ then

$$H^{(2\text{D})}(M(\boldsymbol{z})) = (1 - z_1)^{kl} R_2(\boldsymbol{z})$$

How To Design the Mapping

The mapping has to satisfy

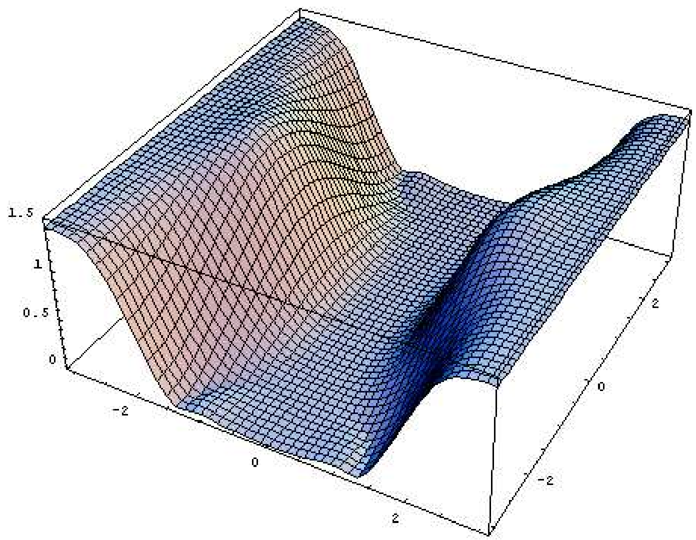
$$M(\boldsymbol{z}) = -M(-\boldsymbol{z}), \quad \text{and}$$
$$M(\boldsymbol{z}) = (1 - z_1)^l m(\boldsymbol{z}) - 1$$

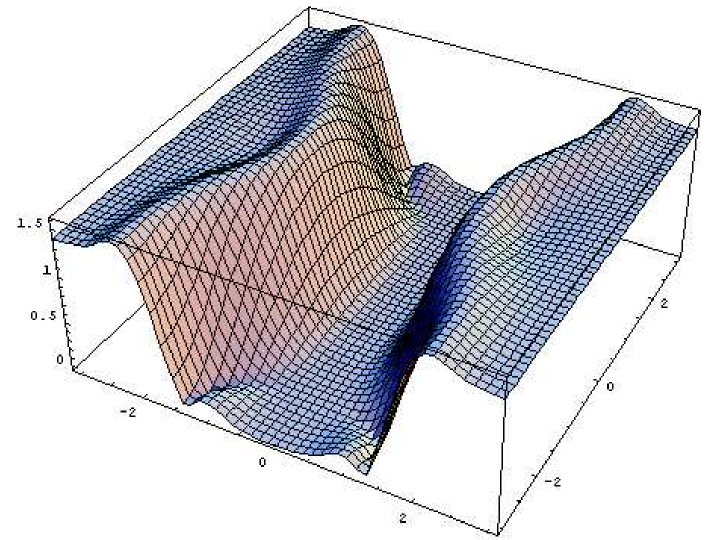
Thus,

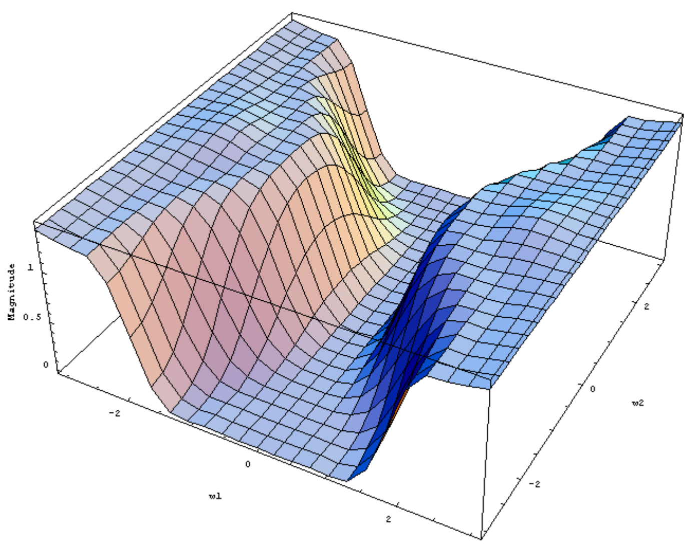
$$(1 - z_1)^l m(\boldsymbol{z}) + (1 + z_1)^l m(-\boldsymbol{z}) = 2$$

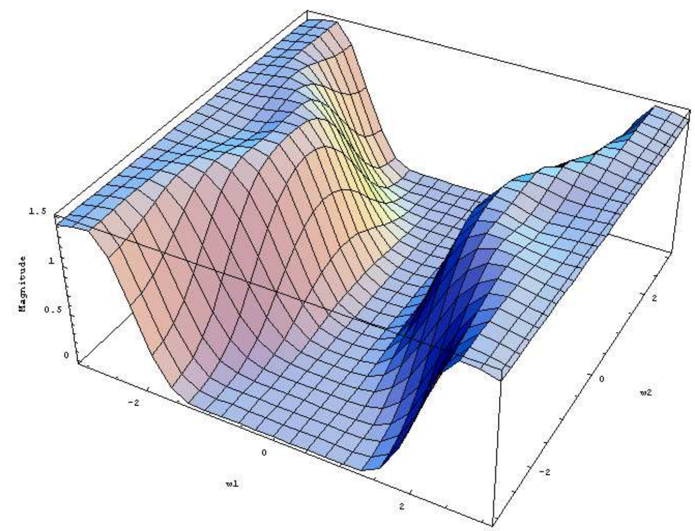
The last proposition tell us exactly how to solve this!

Design Examples



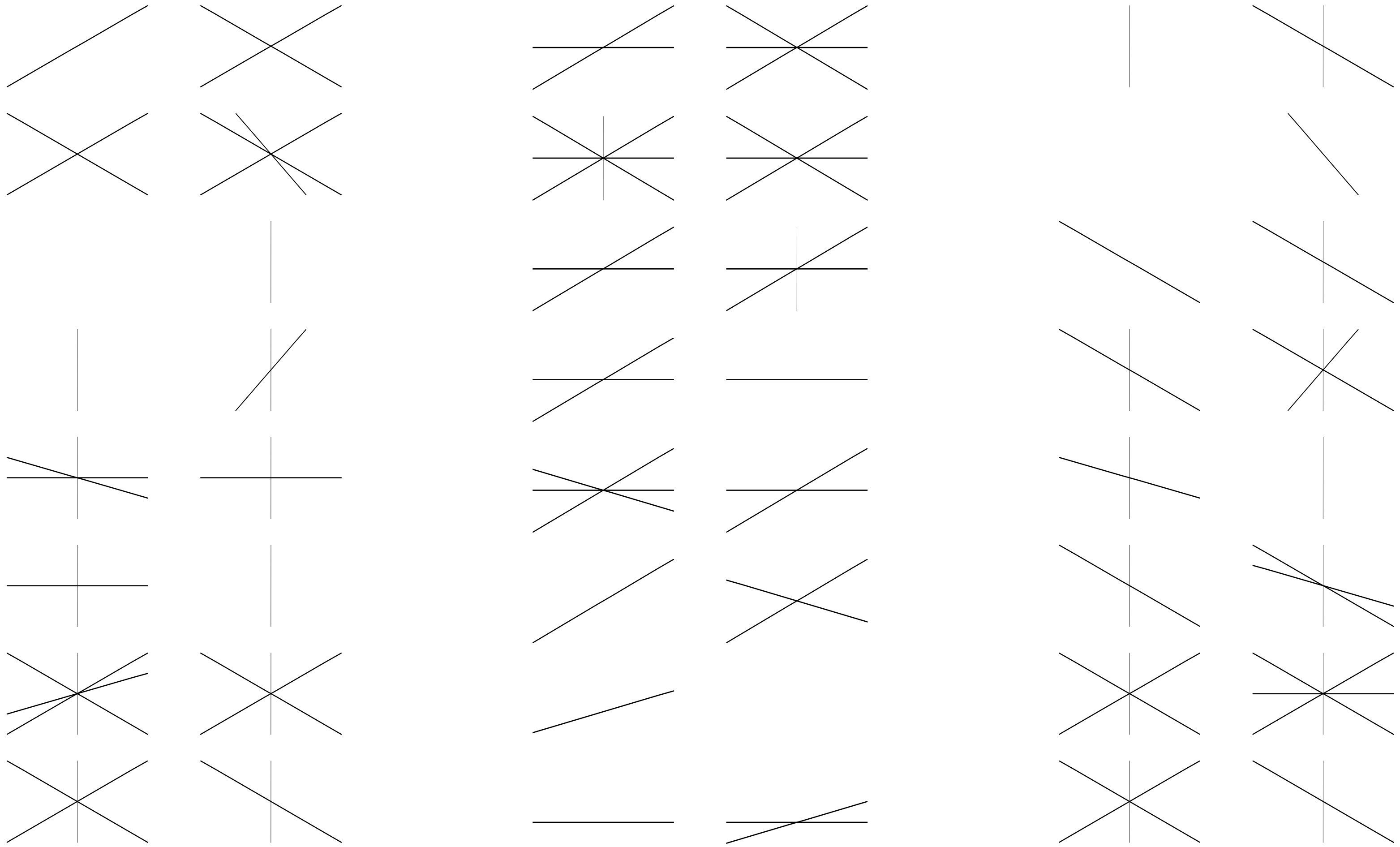




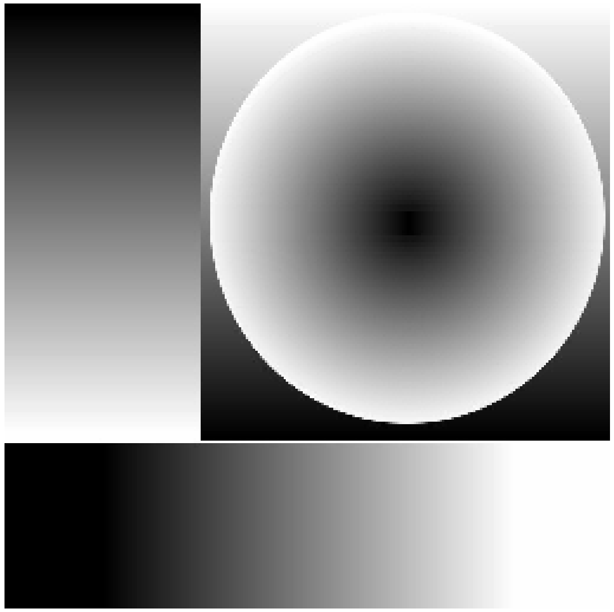


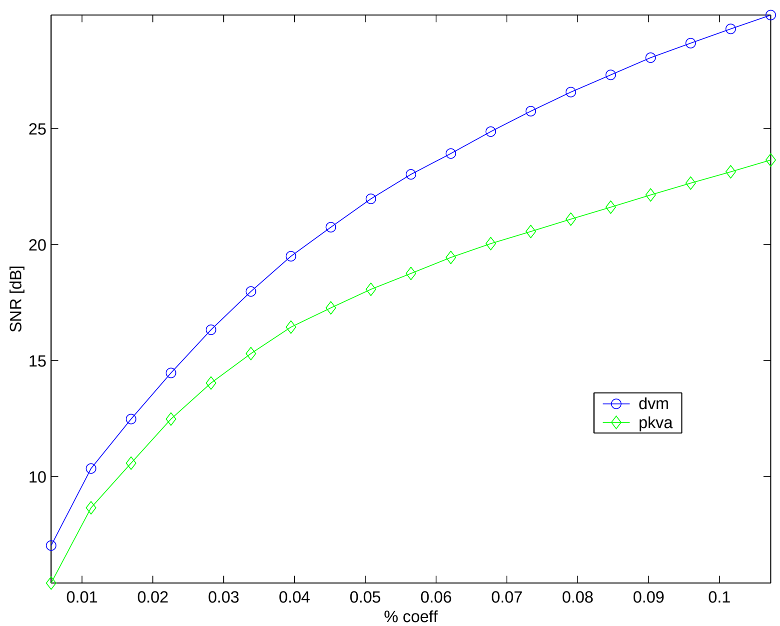
Directional Vanishing Moments Generated After Iteration

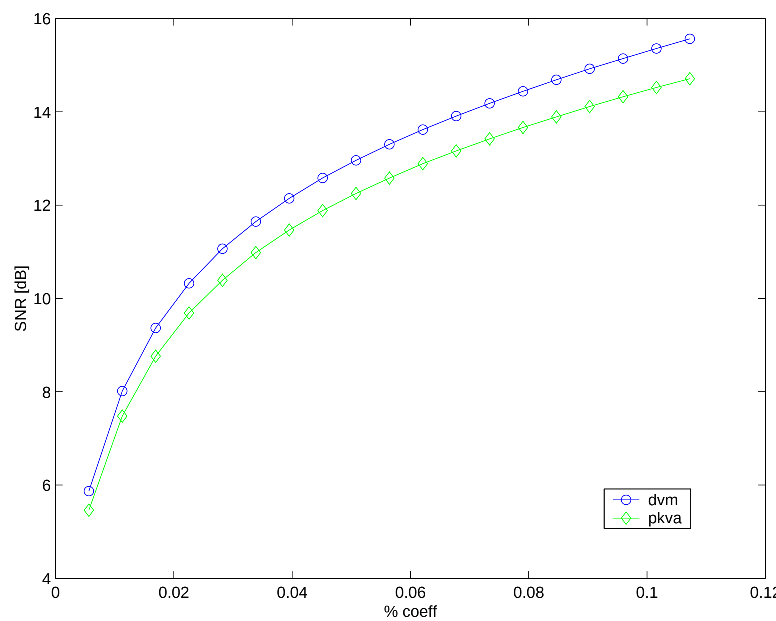
Different expanding rules lead to different set of directions with DVMs.



Gain by using Filters with DVMs







Summary

- Image processing relies on *prior information* about images.
  - Geometrical structure is the key!
- Strong motivation for more powerful image representations: scale, space, and direction.
  - New desideratum beyond wavelets: localized direction
- New two-dimensional discrete framework and algorithms:
  - Flexible directional and multiresolution image representation.
  - Effective for images with smooth contours $\Rightarrow$ contourlets.
- Dream: Another fruitful interaction between harmonic analysis, computer vision, and signal processing.

References

- M. N. Do, "Directional multiresolution image representations," Ph.D. thesis, EPFL, December 2001.
- M. N. Do and M. Vetterli, "Contourlets," *Beyond Wavelets*, G. V. Welland ed., Academic Press, 2003.
- M. N. Do and M. Vetterli, "The contourlet transform: an efficient directional multiresolution image representation," *IEEE Trans. on Image Proc.*, submitted 2003.
- A. Cunha and M. N. Do, "Biorthogonal two-channel filter banks with directional vanishing moments: characterization, design and applications," *IEEE Trans. on Image Proc.*, submitted 2004.

- **Software and downloadable papers**: `www.ifp.uiuc.edu/~minhdo`