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LAURENT POLYNOMIAL INVERSE MATRICES AND MULTIDIMENSIONAL PERFECT RECONSTRUCTION SYSTEMS

BY

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DISSERTATION

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Abstract

We study the invertibility of M-variate polynomial (respectively : Laurent polynomial) matrices of size N by P. Such matrices represent multidimensional systems in various settings including filter banks, multiple-input multiple-output systems, and multirate systems. Given an $N \times P$ polynomial matrix $\boldsymbol{H}(\boldsymbol{z})$ of degree at most k, we want to find a $P \times N$ polynomial (resp. : Laurent polynomial) left inverse matrix $\boldsymbol{G}(\boldsymbol{z})$ of $\boldsymbol{H}(\boldsymbol{z})$ such that $\boldsymbol{G}(\boldsymbol{z})\boldsymbol{H}(\boldsymbol{z}) = \boldsymbol{I}$. We provide computable conditions to test the invertibility and propose algorithms to find a particular inverse. The main result of this thesis is to prove that when $N - P \geq M$, then $\boldsymbol{H}(\boldsymbol{z})$ is generically invertible; whereas when $N - P < M$, then $\boldsymbol{H}(\boldsymbol{z})$ is generically noninvertible. Based on this fact, we provide some applications and propose a faster algorithm to find a particular inverse of a Laurent polynomial matrix.

The next main topic we are interested is the theory and algorithms for the optimal use of multidimensional signal reconstruction from multichannel acquisition using a filter bank setup. Suppose that we have an N-channel convolution system in M dimensions. Instead of taking all the data and applying multichannel deconvolution, we can first reduce the collected data set by an integer $M \times M$ sampling matrix $\boldsymbol{D}$ and still perfectly reconstruct the signal with a synthesis polyphase matrix. First, we determine the existence of perfect reconstruction systems for given finite impulse response (FIR) analysis filters with some sampling matrices and some FIR synthesis polyphase matrices. Second, we present an efficient algorithm to find a sampling matrix with maximum sampling rate and FIR synthesis polyphase matrix for given FIR analysis filters so that the system provides a perfect reconstruction. Third, we develop an algorithm to find a FIR synthesis polyphase matrix for given FIR analysis filters with pure delays allowed in each branch of analysis filters before a given downsampling. Next, once a particular synthesis matrix is found, we can characterize all synthesis matrices and find an optimal one by applying frame analysis and according to design criteria including robust reconstruction in the presence of noise.

Instead of focusing on the application, we are also interested in more theoretical setting. We discuss the conditions on density of the set of invertible (resp. : noninvertible) $N \times P$ matrices. Lastly we study the generalized inverse on polynomial (resp. : Laurent polynomial) matrices.

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List of Symbols

k	Field
$\mathbb{C}$	Complex number
$\mathbb{R}$	Real number
$\mathbb{T}^n$	Power product
$lp(f)$	Leading power product of f
$lc(f)$	Leading coefficient of f
$lt(f)$	Leading term of f
$lm(f)$	Leading monomial of f
$s(f,g)$	S-polynomial of f and g
$f \xrightarrow{g} h$	f reduces to h modulo g
$f \xrightarrow{F}_+ h$	f reduces to h modulo the set F
$\mathrm{RES}_{(k_0,...,k_n)}$	Resultant of fixed positive degrees $k_0, ..., k_n$
$V(F)$	Variety of the set of polynomials F
$H(\boldsymbol{z})$	$N \times P$ matrix over $\mathbb{C}[z_1, ..., z_M]$ or $\mathbb{R}[z_1, ..., z_M]$
λ_n	$2n$-dimensional Lebesgue measure
MSE	Mean square error
MMSE	Minimum mean square error
$\mathrm{ht}I$	The height of ideal I
$\sqrt{I}$	The radical ideal of I
$LAT(D)$	The set of all vectors of the form $\boldsymbol{Dm}$, $\boldsymbol{m} \in \mathbb{Z}^M$
$\mathcal{N}(\boldsymbol{D})$	The quotient group of $\boldsymbol{D}\mathbb{Z}^M$ in $\mathbb{Z}^M$
$\mathrm{adj}(T)$	Adjoint matrix of T
$\det(T)$	Determinant of T
$\gcd(f,g)$	Greatest common divisor of f and g
$\mathrm{lcm}(f,g)$	Least common multiple of f and g

$\mathcal{N}^k_{N,P,M}$	Set of $N \times P$ left noninvertible matrices over $\mathbb{C}[z_1, ..., z_M]$ of degree at most k
$\mathcal{I}^k_{N,P,M}$	Set of $N \times P$ left invertible matrices over $\mathbb{C}[z_1, ..., z_M]$ of degree at most k
$\|\|.\|\|_2$	Two norm
$R(A)$	Range of A
$N(A)$	Null space of A
A^*	Conjugate transpose of A
P_M	Orthogonal projector of $\mathbb{C}^n$ onto M
$A^\dagger$	Generalized inverse of A
$\text{rank}(A)$	Rank of a matrix A
$A^\dagger(z)$	Generalized inverse of $A(z)$

Chapter 1

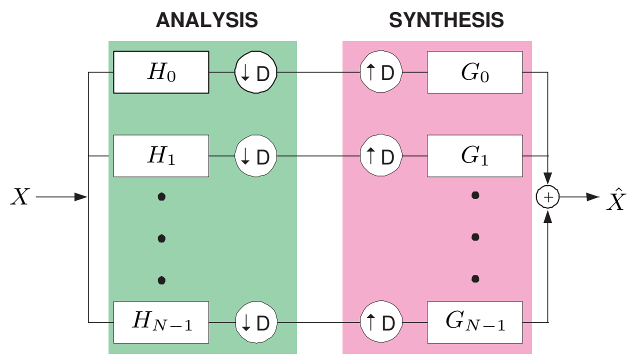
Introduction

1.1 Motivation

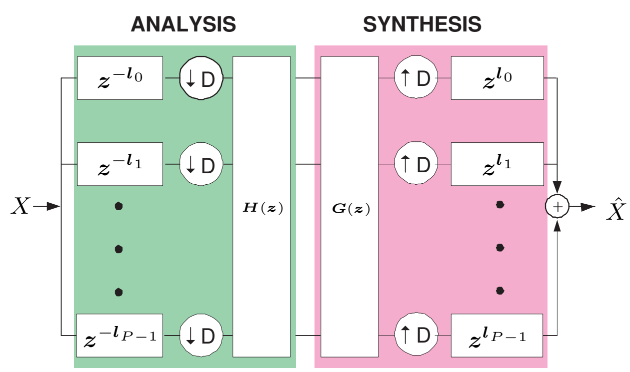
During the last two decades, one dimensional multirate systems in digital signal processing were thoroughly developed. In recent years, due to the high demand in multidimensional signal processing including image and video processing, volumetric data analysis and spectroscopic imaging, multidimensional multirate systems have been studied more extensively. One key property of a multidimensional multirate system is its perfect reconstruction, which guarantees that an original input can be perfectly reconstructed from the outputs.

In a multidimensional multirate system, a digital signal is split into several channels and processed with different sampling rates. The most popular multirate systems are filter banks shown in Fig. 1.1(a). In the analysis part, a digital input signal is filtered and then downsampled, generating multiple outputs at the lower rates. In the synthesis part, the multiple outputs are upsampled and then filtered to reconstruct the original signal. Using the polyphase representation in the z-domain [57, 59], we can represent the analysis part as an $N \times P$ matrix $\boldsymbol{H}(\boldsymbol{z})$ with entries in a Laurent polynomial ring $\mathbb{C}[z_1, z_2, ..., z_M, z_1{}^{-1}, ..., z_M{}^{-1}]$ shown in Fig.1.1(b). Here M is the dimension of signals, N is the number of channels in the filter bank, and P is the sampling factor at each channel. An application of this setting may arise in multichannel acquisition. Here we collect data about unknown multidimensional signal $\boldsymbol{X}(\boldsymbol{z})$ as output of the analysis part in Fig. 1.1(a). The acquisition system (filters $H_i(\boldsymbol{z})$ and sampling matrix $\boldsymbol{D}$) is fixed and known beforehand. The objective is to reconstruct $\boldsymbol{X}(\boldsymbol{z})$ with a synthesis part $\boldsymbol{G}(\boldsymbol{z})$. The existence of a synthesis part becomes a purely mathematical question. Therefore, our first problem is to consider whether there exists a $P \times N$ matrix $\boldsymbol{G}(\boldsymbol{z})$ over a Laurent polynomial ring $\mathbb{C}[z_1, z_2, ..., z_M, z_1{}^{-1}, ..., z_M{}^{-1}]$ for which $\boldsymbol{G}(\boldsymbol{z})$ satisfies $\boldsymbol{G}(\boldsymbol{z})\boldsymbol{H}(\boldsymbol{z}) = \boldsymbol{I}_P$ where $\boldsymbol{I}_P$ is the $P \times P$ identity matrix?

One dimensional perfect reconstruction finite impulse response (FIR) filter banks have been investigated in several studies [6, 16, 34]. The Euclidean algorithm plays a key role in the matrix inverse problem for one dimensional perfect reconstruction FIR filter banks [16]. However, there is no Euclidean algorithm for multivariate polynomials. Therefore, the theory of Gröbner bases has been introduced to compute



(a)



(b)

Figure 1.1: Example system represented by a polynomial matrix. (a) A multidimensional N-channel oversampled filter bank: H_i and G_i are analysis and synthesis filters, respectively; $\boldsymbol{D}$ is an $M \times M$ sampling matrix with sampling rate $P = |\det \boldsymbol{D}| \leq N$. (b) Polyphase representation: $\boldsymbol{H}(\boldsymbol{z})$ and $\boldsymbol{G}(\boldsymbol{z})$ are analysis and synthesis polyphase transformation matrices, respectively; $\{\boldsymbol{l}_i\}$ is a basis of the lattice generated by the sampling matrix $\boldsymbol{D}$.

with multivariate polynomials [1, 15] and are widely used in multidimensional signal processing [12, 11, 45, 44]. Methods for testing the invertibility and for computing a particular inverse of an $N \times 1$ multivariate polynomial (resp. : Laurent polynomial) matrix $\boldsymbol{H}(\boldsymbol{z})$ were proposed in [49, 63] by using the technique of Gröbner bases. One technique to apply these methods to a general $N \times P$ matrix $\boldsymbol{H}(\boldsymbol{z})$ is to consider the maximal minors and their corresponding adjoint matrices [49, 48]. Alternatively, Park in [42] provides a method to compute an $P \times N$ inverse Laurent polynomial matrix $\boldsymbol{G}(\boldsymbol{z})$. His method involves transforming Laurent polynomials into polynomials by multiplying a series of elementary matrices. In Chapter 3, we ofter a simpler and more direct algorithm to compute a particular Laurent polynomial inverse.

The second question is: When does the system have a high probability of the existence of an inverse? Rajagopal and Potter [49] and Zhou and Do [62] have investigated this question and made several conjectures. In Chapter 4, we investigate the systems by varying M, N and P. In the experiments, we found that when $M - N \geq P$, the existence of an inverse is "almost surely". On the other hand, when $M - N < P$, the nonexistence of an inverse is "almost surely". To precisely study this inverse existence problem, we employ the measure theory [51] and the concept of "hold generically" [15]. Then we will talk about some applications on generic invertibility.

Consider a known N-channel convolution system in M dimensions. One would like to know what the condition is such that we can apply a perfect reconstruction deconvolution to reconstruct the original signal. Then instead of taking all the data and apply multichannel deconvolution, we can minimize the collected data set by a sampling matrix $\boldsymbol{D}$ and with the reduced data can still apply a perfect reconstruction synthesis filters to reconstruct the signal. In Chapter 5, we will discuss how to reduce collected data set after employing Hermite and Smith norm forms. We can then generate all inverses from a particular inverse. In this set of inverses, one find an optimal set of synthesis filters according to some design criteria.

In Chapter 6, we study the conditions such that the set of invertible (resp. : noninvertible) $N \times P$ matrices to be dense.

Now we turn to study the generalized inverse. In Chapter 7, we extend the concept of the generalized inverse from matrices with complex numbers to polynomials (resp. : Laurent polynomial).

1.2 Outline of the Document

In this work, we are interested in the invertibility of a matrix. The outline of this thesis is the following:

- Overview of Gröbner Bases and the software **Singular**.

- Computational Issue: How can we determine a matrix invertible over a polynomial ring or a Laurent polynomial ring? Can we design an inverse algorithm?

- Generic invertibility: We will show an $N \times P$ matrix in M variables is generic invertible if $N \times P \geq M$; whereas an $N \times P$ matrix in M variables is generic noninvertible if $N \times P < M$.

- Application on generic invertibility.

- Can we find an algorithm to synthesis filters and a sampling matrix such that the system is perfect reconstruction and the collected data is the minimum? Can we find optimal inverses such that the system by minimizing different norms?

- Density: What condition would the set of left invertible $N \times P$ matrices over a polynomial ring to be dense? What condition would the set of left non-invertible $N \times P$ matrices over a polynomial ring or a Laurent polynomial ring to be not dense?

- Generalized Inverses: What is a generalized inverse over a polynomial ring or a Laurent polynomial ring?

Chapter 2

Basic Tools

2.1 An Introduction to Gröbner Bases

The following section is adopted from [1].

The Hilbert's basis theorem is proved by David Hilbert in 1880, which states that every ideal in multivariate polynomials ring $k[x_1, ..., x_n]$ is finitely generated. Nevertheless, Hilbert did not provide a constructive way to find a basis of a given ideal. In 1965, Bruno Buchberger develop the theory of Gröbner Bases for polynomial rings. The theory can be expressed as a generalization of the theory of polynomials in one variable.

Nowadays, Gröbner Bases computation is an essential tool in computer algebra, computation algebraic geometry and commutative algebra. There are many commercial or free computer algebra software to implement the computation of Gröbner Bases.

For more information, please refer to [1, 2, 14, 19].

Definition 1. *Given $k[x_1, x_2, ..., x_n]$, we denote a* power product *(or* monomial*)*

$$\mathbb{T}^n = \{x_1^{\beta_1}...x_n^{\beta_n} | \beta_i \in \mathbb{N}, i = 1, .., n\}.$$

Also we denote a term *to be a coefficient times a power product.*

Definition 2. *We define the* lexicographical order *on $\mathbb{T}^n$ with $x_1 > x_2 > ... > x_n$ as follows: For*

$$\alpha = (\alpha_1, ..., \alpha_n),\ \beta = (\beta_1, ..., \beta_n) \in \mathbb{N}^n$$

we define $x^\alpha < x^\beta$ if the first nonzero coordinate in $\beta - \alpha$ is positive.

Definition 3. *We define the* degree lexicographical order *on $\mathbb{T}^n$ with $x_1 > x_2 > ... > x_n$ as follows: For*

$$\alpha = (\alpha_1, ..., \alpha_n), \beta = (\beta_1, ..., \beta_n) \in \mathbb{N}^n,$$

we define $x^\alpha < x^\beta$ *to be either* $\sum_{i=1}^n \alpha_i < \sum_{i=1}^n \beta_i$ *or* $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i$ *and* $x^\alpha < x^\beta$ *with respect to lex with* $x_1 > x_2 > ... > x_n$.

Definition 4. *For a fixed term order, any* $f \in k[x_1, ..., x_n]$ *with* $f \neq 0$*, we may write*

$$f = a_1 x^{\alpha_1} + a_2 x^{\alpha_2} + ... + a_r x^{\alpha_r}$$

where $x^{\alpha_i} \in \mathbb{T}^n$*, and* $x^{\alpha_1} > x^{\alpha_2} > ... > x^{\alpha_r}$*. We define*

- $lp(f) = x^{\alpha_1}$*, the* leading power product *of* f*;*
- $lc(f) = a_1$*, the* leading coefficient *of* f*;*
- $lt(f) = a_1 x^{\alpha_1}$*, the* leading term *of* f*.*

Definition 5. *Let* $f, g \in k[x_1, x_2, ..., x_n]$ *with* $f, g \neq 0$*. Let* $L = \mathrm{lcm}(lp(f), lp(g))$*. The polynomial*

$$S(f, g) = \frac{L}{lt(f)} f - \frac{L}{lt(g)} g$$

is called the S-polynomial *of* f *and* g*.*

Definition 6. *Given* f, g, h *in* $k[x_1, ..., x_n]$*, with* $g \neq 0$*, we say that* f reduce to h modulo g *in one step, written*

$$f \xrightarrow{g} h$$

if and only if $lp(g)$ *divides a non-zero term* x^α *that appears in* f *and*

$$h = f - \frac{x^\alpha}{lt(g)} g.$$

Now let f, h *and* $f_1, ..., f_s$ *be polynomials in* $k[x_1, ..., x_n]$*, with* $f_i \neq 0$ *for* $i = 1, ..., s$*, and let* $F = \{f_1, ..., f_s\}$*. We say that* f reduces to h modulo F*, denoted*

$$f \xrightarrow{F}_+ h$$

if and only if there exist a sequence of indices $i_1, i_2, ..., i_t \in \{1, ..., s\}$ *and a sequence of polynomials* $h_1, ..., h_{t-1} \in k[x_1, ..., x_n]$ *such that*

$$f \xrightarrow{f_{i_1}} h_1 \xrightarrow{f_{i_2}} h_2 \xrightarrow{f_{i_3}} ... \xrightarrow{f_{i_{t-1}}} h_{t-1} \xrightarrow{f_{i_t}} h.$$

Definition 7. *A polynomial r is called* reduced *with respect to a set of non-zero polynomials $F = \{f_1, ..., f_s\}$ if $r = 0$ or no power product that appears in r is divisible by any one of the $lp(f_i)$, $i = 1, ..., s$. In other words, r cannot be reduced modulo F.*

Definition 8. *A set of non-zero polynomials $G = \{g_1, ..., g_t\}$ contained in an ideal I, is called a* Gröbner basis *for I if and only if for all $f \in I$ such that $f \neq 0$, there exists $i \in \{1, ..., t\}$ such that $lp(g_i)$ divides $lp(f)$.*

Definition 9. *A Gröbner basis $G = \{g_1, ..., g_t\}$ is called* minimal *if for i, $lc(g_i) = 1$ and for $i \neq j$, $lp(g_i)$ does not divide $lp(g_j)$.*

Definition 10. *A Gröbner basis $G = \{g_1, ..., g_t\}$ is called* reduced Gröbner basis *if, for all i, $lc(g_i) = 1$ and g_i is reduced with respect to $G - \{g_i\}$. That is, for all i, no non-zero term in g_i is divisible by any $lp(g_j)$ for any $j \neq i$.*

Theorem 1. (**Buchberger**) *[1, p.40] Let $G = \{g_1, ..., g_t\}$ be a set of non-zero polynomials in $k[x_1, ..., x_n]$. Then G is a Gröbner basis for the ideal $I = \langle g_1, ..., g_t \rangle$ if and only if for all $i \neq j$,*

$$S(g_i, g_j) \xrightarrow{G}_+ 0.$$

Algorithm 1 (Buchbeger's Algorithm)**.** *[1, p.43] The Algorithm for Computing Gröbner Basis of $\langle f_1, ..., f_s \rangle$.*
Input: $F = \{f_1, ..., f_s\} \subset k[x_1, ..., x_n]$ with $f_i \neq 0$ $(1 \leq i \leq s)$.
Output: $G = \{g_1, ..., g_t\}$, a Gröbner basis for $\langle f_1, ..., f_s \rangle$.
Set $G := F, \mathcal{G} = \{\{f_i, f_j\} | f_i \neq f_j \in G\}$.
1. If $\mathcal{G} = \emptyset$, then output G.
2. Choose any $\{f, g\} \in \mathcal{G}$.
3. $\mathcal{G} := \mathcal{G} - \{f, g\}$.
4. $S(f, g) \xrightarrow{G}_+ h$ where h is reduced with respect to G.
5. If $h \neq 0$, then$\mathcal{G} := \mathcal{G} \cup \{\{u, h\}$for all $u \in G\}$ and $G := G \cup \{h\}$.
6. Go to 1.

Theorem 2. *[1, p.48] Fix a term order. Then every nonzero ideal I has a unique reduced Gröbner basis with respect to this term order.*

Theorem 3. *[1, p.63] and [2, p.274] The following statements are equivalent.*
(1) *The variety $V(I)$ is finite;*
(2) *For every term order $\leq$ on $\mathbb{T}^n$ and every Gröbner basis G of I with respect to $\leq$, for all $i = 1, ..., n$, there exists $j \in \{1..., t\}$ such that $lp(g_j) = x_i^v$ for some $v \in \mathbb{N}$.*

Lemma 1. *[1, p.125] Let $f, g \in k[x_1, ..., x_n]$, both non-zero, and let $d = \gcd(f, g)$. The following statements are equivalent:*

(1) *$lp(\frac{f}{d})$ and $lp(\frac{g}{d})$ are relatively prime;*

(2) *$S(f, g) \xrightarrow{\{f,g\}}_+ 0$.*

where lp is the leading power product of f (i.e. the greatest sum of the powers among the terms) and gcd is the greatest common divisor.

In particular, $\{f, g\}$ is a Gröbner basis if and only if $lp(\frac{f}{d})$ and $lp(\frac{g}{d})$ are relatively prime.

2.2 Introduction to Singular: Gröbner Basis Computation

Singular is a computer algebra system which can compute a Gröbner basis for a given set of polynomials. [23] is excellent reference for more information on Gröbner bases and their applications for **Singular**.

Example 1. *To calculate a Gröbner basis, a ring has to be defined first:*

`>ring R=(real,20), (x,y), dp;`

where R is a ring with 2 variables and real floating point numbers, 20 digits precision. The `dp` *at the end means that the degree reverse lexicographical ordering is used.*

To calculate a Gröbner basis for a given set of polynomials:

`>ideal I= x2+xy2+x, 2+y2, x-1;`

`>ideal J=std(I);` `std` *command returns a Gröbner basis of I.*

```
>print(J);
x-1
y2+2
```

2.3 Gröbner Bases for Modules

Definition 11. *Let R be a commutative ring. An R-*module *is an abelian group M equipped with a scalar multiplication $R \times M \to M$, denoted by*

$$(r, m) \mapsto rm,$$

such that the following axioms hold for all $m, m' \in M$ and all $r, r', 1 \in R$:

1. $r(m + m') = rm + rm'$;

2. $(r + r')m = rm + rm'$;

3. $(rr')m = r(r'm)$;

4. $1m = m$.

Example 2. *Let* $R = k[x_1, ..., x_n]$. *Then* R^m *is R-module, where scalar multiplication* $R \times R^m \to R^m$ *is the given multiplication* $(r, (r_1, ..., r_m)) \mapsto (rr_1, ..., rr_m)$.

Definition 12. *If* M *is an R-module, then a* submodule N *of* M*, denoted by* $N \subset M$*, is an additive subgroup* N *of* M *closed under scalar multiplication:* $rn \in N$ *whenever* $n \in N$ *and* $r \in R$.

Example 3. *If* M *is an R-module and* X *is a subset of an R-module* M*, then*

$$\langle X \rangle = \{ \sum_{finite} r_i x_i : r_i \in R \text{ and } x_i \in X \}$$

is a submodule of M.

In the following discussion, we let $R = k[x_1, ..., x_n]$.Now, we will generalize the theory of Gröbner Bases to submodules of R^m. As a result we will be able to compute with submodules of R^m in a way similar to the way we computed with ideals previously. Let

$$e_1 = (1, 0, ..., 0),\ e_2 = (0, 1, 0, ..., 0),\ ...,\ e_m = (0, ..., 0, 1),$$

of R^m be the standard basis.

Definition 13. *We denote a* monomial *in* R^m *by a vector of the type* Xe_i $(1 \leq i \leq m)$*, where* X *is a power product in* R*. If* $\mathbf{X} = Xe_i$ *and* $\mathbf{Y} = Ye_j$ *are monomials in* R^m*, we say that* $\mathbf{X}$ *divides* $\mathbf{Y}$ *provided that* $i = j$ *and* X *divides* Y.

Also we denote a term *by a vector of the type* $c\mathbf{X}$*, where* $c \in k - \{0\}$ *and* $\mathbf{X}$ *is a monomial. If* $\mathbf{X} = cXe_i$ *and* $\mathbf{Y} = dYe_j$ *are terms of* R^m*, we say* $\mathbf{X}$ *divides* $\mathbf{Y}$ *provided that* $i = j$ *and* X *divides* Y*. We write*

$$\frac{\mathbf{X}}{\mathbf{Y}} = \frac{cX}{dY}.$$

Definition 14. *By a* term order *on the monomials of* R^m *we mean a total order,* $<$*, on these monomials satisfying the following two conditions:*

1. $\mathbf{X} < Z\mathbf{X}$*, for every monomial* $\mathbf{X}$ *of* R^m *and power product* $Z \neq 1$ *of* R*;*

2. If $\mathbf{X} < \mathbf{Y}$*, then* $Z\mathbf{X} < Z\mathbf{Y}$ *for all monomials* $\mathbf{X}, \mathbf{Y} \in R^m$ *and every power power* $Z \in R$.

Definition 15. *For monomials* $\mathbf{X} = Xe_i$ *and* $\mathbf{Y} = Ye_j$ *of* R^m*, we say that*

$$\mathbf{X} < \mathbf{Y} \iff \begin{cases} X < Y \\ \text{or} \\ X = Y \quad \text{and } i < j. \end{cases}$$

We call this order TOP *for "term over position".*

Definition 16. *For monomials* $\mathbf{X} = Xe_i$ *and* $\mathbf{Y} = Ye_j$ *of* R^m*, we say that*

$$\mathbf{X} < \mathbf{Y} \iff \begin{cases} i < j \\ \text{or} \\ i = j \quad \text{and } X < Y. \end{cases}$$

We call this order POT *for "position over term".*

Definition 17. *Now for a fixed term order* $<$ *on the monomials of* R^m*. Any* $\mathbf{f} \in R^m$*, with* $\mathbf{f} \neq \mathbf{0}$*, we may write*

$$\mathbf{f} = a_1\mathbf{X_1} + a_2\mathbf{X_2} + ... + a_r\mathbf{X_r},$$

where, for $1 \leq i \leq r, 0 \neq a_i \in k$ *and* $\mathbf{X_i}$ *is a monomial in* R^m *satisfying* $\mathbf{X_1} > \mathbf{X_2} > ... > \mathbf{X_r}$*. we define*

- $lm(\mathbf{f}) = \mathbf{X_1}$*, the* leading monomial *of* $\mathbf{f}$*;*
- $lc(\mathbf{f}) = a_1$*, the* leading coefficient *of* $\mathbf{f}$*;*
- $lt(\mathbf{f}) = a_1\mathbf{X_1}$*, the* leading term *of* $\mathbf{f}$*.*

We define $lm(\mathbf{0}) = \mathbf{0}, lc(\mathbf{0}) = 0, lt(\mathbf{0}) = \mathbf{0}$*.*

Definition 18. *Let* $\mathbf{f}, \mathbf{g} \in R^m$ *with* $\mathbf{f}, \mathbf{g} \neq \mathbf{0}$*. Let* $\mathbf{L} = \mathrm{lcm}(lm(\mathbf{f}), lm(\mathbf{g}))$*. The vector*

$$S(\mathbf{f}, \mathbf{g}) = \frac{\mathbf{L}}{lt(\mathbf{f})}\mathbf{f} - \frac{\mathbf{L}}{lt(\mathbf{g})}\mathbf{g}$$

is called the S-polynomial *of* $\mathbf{f}$ *and* $\mathbf{g}$*.*

Definition 19. *Given* $\mathbf{f}, \mathbf{g}, \mathbf{h}$ *in* R^m*, with* $\mathbf{g} \neq \mathbf{0}$*, we say that* $\mathbf{f}$ reduce to $\mathbf{h}$ modulo $\mathbf{g}$ *in one step, written*

$$\mathbf{f} \xrightarrow{\mathbf{g}} \mathbf{h}$$

if and only if $lm(\mathbf{g})$ *divides a term* $\mathbf{X}$ *that appears in* $\mathbf{f}$ *and*

$$\mathbf{h} = \mathbf{f} - \frac{\mathbf{X}}{lt(\mathbf{g})}\mathbf{g}.$$

Now let $\mathbf{f}, \mathbf{h}$ *and* $\mathbf{f_1}, ..., \mathbf{f_s}$ *be vectors in* R^m*, with* $\mathbf{f_i} \neq \mathbf{0}$ *for* $i = 1, ..., s$*, and let* $F = \{\mathbf{f_1}, ..., \mathbf{f_s}\}$*. We say that* $\mathbf{f}$ reduces to $\mathbf{h}$ modulo F*, denoted*

$$\mathbf{f} \xrightarrow{F}_{+} \mathbf{h}$$

if and only if there exist a sequence of indices $i_1, i_2, ..., i_t \in \{1, ..., s\}$ *and a sequence of vectors* $\mathbf{h_1}, ..., \mathbf{h_{t-1}} \in R^m$ *such that*

$$\mathbf{f} \xrightarrow{\mathbf{f_{i_1}}} \mathbf{h_1} \xrightarrow{\mathbf{f_{i_2}}} \mathbf{h_2} \xrightarrow{\mathbf{f_{i_3}}} ... \xrightarrow{\mathbf{f_{i_{t-1}}}} \mathbf{h_{t-1}} \xrightarrow{\mathbf{f_{i_t}}} \mathbf{h}.$$

Definition 20. *A vector* r *is called* reduced *with respect to a set* $F = \{\mathbf{f_1}, ..., \mathbf{f_s}\}$ *of non-zero vectors in* R^m *if* $\mathbf{r} = \mathbf{0}$ *or no power product that appears in* $\mathbf{r}$ *is divisible by any one of the* $lm(\mathbf{f_i})$*,* $i = 1, ..., s$*. In other words,* $\mathbf{r}$ *cannot be reduced modulo* F*.*

Definition 21. *A set of non-zero vectors* $G = \{\mathbf{g_1}, ..., \mathbf{g_t}\}$ *contained in the submodule* M *is called a* Gröbner basis *for* M *if and only if for all* $\mathbf{f} \in M$*, there exists* $i \in \{1, ..., t\}$ *such that* $lm(\mathbf{g_i})$ *divides* $lm(\mathbf{f})$*.*

Definition 22. *A Gröbner basis* $G = \{\mathbf{g_1}, ..., \mathbf{g_t}\} \subset R^m$ *is called* reduced Gröbner basis *if, for all* i*,* $lc(\mathbf{g_i}) = 1$ *and* $\mathbf{g_i}$ *is reduced with respect to* $G - \{\mathbf{g_i}\}$*. That is, for all* i*, no non-zero term in* $\mathbf{g_i}$ *is divisible by any* $lm(\mathbf{g_j})$ *for any* $j \neq i$*.*

Theorem 4. *[1, p.148] Let* $G = \{\mathbf{g_1}, ..., \mathbf{g_t}\}$ *be a set of non-zero vectors in* R^m*. Then* G *is a Gröbner basis for the submodule* $M = \langle \mathbf{g_1}, ..., \mathbf{g_t} \rangle$ *if and only if for all* $i \neq j$*,*

$$S(\mathbf{g_i}, \mathbf{g_j}) \xrightarrow{G}_{+} \mathbf{0}.$$

Algorithm 2 (Buchbeger's Algorithm for Modules)**.** *[1, p.149] The Algorithm for Computing Gröbner Basis of* $\langle \mathbf{f_1}, ..., \mathbf{f_s} \rangle$*.*

Input: $F = \{\mathbf{f_1}, ..., \mathbf{f_s}\} \subset R^m$ *with* $\mathbf{f_i} \neq 0$ $(1 \leq i \leq s)$*.*

Output: $G = \{\mathbf{g_1}, ..., \mathbf{g_t}\}$*, a Gröbner basis for* $\langle \mathbf{f_1}, ..., \mathbf{f_s} \rangle$*.*

Set $G := F, \mathcal{G} = \{\{\mathbf{f_i}, \mathbf{f_j}\} | \mathbf{f_i} \neq \mathbf{f_j} \in G\}$*.*

1. If $\mathcal{G} = \emptyset$*, then output* G*.*

2. Choose any $\{\mathbf{f}, \mathbf{g}\} \in \mathcal{G}$*.*

3. $\mathcal{G} := \mathcal{G} - \{\mathbf{f}, \mathbf{g}\}$*.*

4. $S(\mathbf{f},\mathbf{g})\xrightarrow{G}_+\mathbf{h}$ *where* h *is reduced with respect to* G.

5. *If* $\mathbf{h} \neq 0$, *then* $\mathcal{G} := \mathcal{G} \cup \{\{\mathbf{u},\mathbf{h}\}$ *for all* $\mathbf{u} \in G\}$ *and* $G := G \cup \{\mathbf{h}\}$.

6. *Go to 1.*

Example 4. *To calculate a Gröbner basis for a module in* **Singular***:*

```
>module M=[x2+y+2,z+3],[z+x+2,zy+x],[2x+1,1];

>module N=std(M);

>print(N);

2x+1,-4y-9,2y+z+6,4y2z+9yz-4z2-4y-20z-30,

1,2x-4z-13,yz+2z+6,-2y-z-6
```

Chapter 3

Multidimensional Perfect Reconstruction Systems

In Section 3.1, we show how to verify the invertibility of a matrix. In Section 3.2, we propose algorithms to find a particular inverse based on the Gröbner bases computation. Next, we characterize the set of all inverses and find an optimal inverse according to the design criterion.

3.1 Mathematical Contexts

3.1.1 (Left) Inverse Polynomial Matrix Problem

We use boldface letters to denote vectors, or matrices. Let $\boldsymbol{z}$ be an M-dimensional complex variable $\boldsymbol{z} = (z_1, ..., z_M)$ in $\mathbb{C}^M$. For $\boldsymbol{n} = (n_1, ..., n_M) \in \mathbb{Z}^M$, we define the monomial $\boldsymbol{z}^{\boldsymbol{n}} = \prod_{i=1}^{M} z_i^{n_i}$. In this thesis, we will always assume that N, P, and M are positive integers.

Definition 23 (Polynomial or Laurent Polynomial Matrix)**.** *An $N \times P$ matrix $\boldsymbol{H}(\boldsymbol{z})$ is said to be a* polynomial matrix *(resp. :* Laurent polynomial matrix*) if every entry is a polynomial (resp. : Laurent polynomial).*

Definition 24 (Left Invertible)**.** *An $N \times P$ polynomial (resp. : Laurent polynomial) matrix $\boldsymbol{H}(\boldsymbol{z})$ is said to be* polynomial (resp. : Laurent polynomial) left invertible *if there exists a $P \times N$ polynomial (resp. : Laurent polynomial) matrix $\boldsymbol{G}(\boldsymbol{z})$ such that*

$$\boldsymbol{G}(\boldsymbol{z})\boldsymbol{H}(\boldsymbol{z}) = \boldsymbol{I}_P. \tag{3.1}$$

Otherwise $\boldsymbol{H}(\boldsymbol{z})$ is said to be polynomial (resp. : Laurent polynomial) left noninvertible *.*

The discussion of polynomial (resp. : Laurent polynomial) left invertible can also apply to polynomial (resp. : Laurent polynomial) right invertible. To avoid repetition, throughout the thesis we use the words "invertible" to represent either polynomial left invertible or Laurent polynomial left invertible. It will be clear in the context whether it is polynomial left invertible or Laurent polynomial left invertible.

Consider an $N \times 1$ matrix $H(z)$ over $\mathbb{C}[z]$ where $H_i(z)$ is the i-th row of $H(z)$. If the greatest common divisor (GCD) of $\{H_1(z), ..., H_N(z)\}$ is 1, then the Bezout identity problem has a solution [3]. We can use the Euclidean algorithm to find the GCD and also a set of $\{G_1(z), ..., G_N(z)\}$ [4] such that

$$\sum_{j=1}^{N} G_j(z)H_j(z) = 1.$$

However, the univariate GCD criterion and Euclidean algorithm fail for multivariate polynomials. But the multivariate membership problem can be solved by using Gröbner bases [1, 2]. Briefly, the theory of Gröbner bases guarantees that any set of generators has a unique *reduced Gröbner basis* for a given ordering by using Buchberger's algorithm [7]. If $\{\boldsymbol{b}_1(\boldsymbol{z}), ..., \boldsymbol{b}_n(\boldsymbol{z})\}$ is a Gröbner basis of the $\mathbb{C}[\boldsymbol{z}]$-submodule spanned by the rows of $\boldsymbol{H}(\boldsymbol{z})$, then there exists $n \times N$ transformation matrix $\boldsymbol{W}(\boldsymbol{z})$ such that

$$\begin{pmatrix} \boldsymbol{b}_1(\boldsymbol{z}) \\ \vdots \\ \boldsymbol{b}_n(\boldsymbol{z}) \end{pmatrix} = \boldsymbol{W}(\boldsymbol{z})\boldsymbol{H}(\boldsymbol{z}). \tag{3.2}$$

The important of the Buchberger's algorithm is that the computations of Gröbner bases are available in most of computer algebra software such as Singular, Macauley2, Maple, and Mathematica.

3.1.2 Criteria for Left Invertibility

We can generalize Proposition 2 from [63], which considers the case $P = 1$, so that we can determine whether an $N \times P$ polynomial matrix is invertible or not.

Proposition 1. *Suppose $\boldsymbol{H}(\boldsymbol{z})$ is an $N \times P$ polynomial matrix. Let $S = \langle \boldsymbol{h}_1(\boldsymbol{z}), ..., \boldsymbol{h}_N(\boldsymbol{z})\rangle$ be the $\mathbb{C}[\boldsymbol{z}]$-submodule of $\mathbb{C}[\boldsymbol{z}]^P$ generated by the rows $\boldsymbol{h}_i(\boldsymbol{z})$ of $\boldsymbol{H}(\boldsymbol{z})$. Then $\boldsymbol{H}(\boldsymbol{z})$ is invertible if and only if the reduced Gröbner basis of S is $\{\boldsymbol{e}_i\}_{i=1,...,P}$ where $\boldsymbol{e}_i$ is the i-th row of the $P \times P$ identity matrix.*

Proof. Suppose $\boldsymbol{H}(\boldsymbol{z})$ is invertible. Then there exist $\boldsymbol{G}(\boldsymbol{z}) = (g_{ij}(\boldsymbol{z}))$ such that satisfying (3.1). Then

$$\boldsymbol{e}_i = \sum_{j=1}^{N} g_{ij}(\boldsymbol{z})\boldsymbol{h}_j(\boldsymbol{z}) \tag{3.3}$$

for $i = 1, ..., P$. According to the definition of Gröbner basis [1, p.121], $\{\boldsymbol{e}_i\}_{i=1,..,P}$ is a Gröbner basis of S. It is a reduced Gröbner basis since $\boldsymbol{e}_i$ are linearly independent. By the uniqueness of reduced Gröbner basis with respect to a given term order, $\{\boldsymbol{e}_i\}_{i=1,..,P}$ is the reduced Gröbner basis of S.

Suppose the reduced Gröbner basis of S is $\{\boldsymbol{e}_i\}_{i=1,..,P}$. Then there exist some $\{g_{ij}(\boldsymbol{z})\}$ satisfying (3.3). Let $\boldsymbol{G}(\boldsymbol{z}) = (g_{ij}(\boldsymbol{z}))$. Then

$$\boldsymbol{G}(\boldsymbol{z})\boldsymbol{H}(\boldsymbol{z}) = \boldsymbol{I}.$$

Thus $\boldsymbol{H}(\boldsymbol{z})$ is invertible. □

Example 5. *Is* $\boldsymbol{H}(z_1, z_2) = \begin{pmatrix} 1 & 3z_2 \\ 2z_1+1 & 0 \\ 3 & z_1 \\ 3z_2 & 5 \end{pmatrix}$ *invertible? We can use the software* **Singular** *[23] to implement the above result.*

`>ring R=0,(z(1),z(2)),dp;` *% R is a ring with 2 variables;* `dp` *specifies the degree reverse lexicographical ordering.*

`>matrix H[4][2]=1,3*z(2),2*z(1)+1,0,3,z(1),3*z(2),5;`

`>print(H);`

```
1,3*z(2),
2*z(1)+1,0,
3,z(1),
3*z(2),5
```

`>module S=transpose(H);` *% S is the module generated by rows of* $\boldsymbol{H}(z_1, z_2)$*.*

`>option(redSB);` *% Computes a reduced standard basis in any standard basis computation.*

`>print(std(S));` *% Returns the reduced Groebner basis by using above option*

```
1,0,
0,1
```

By Proposition 1, we know that $\boldsymbol{H}(z_1, z_2)$ *is invertible.*

The results from algebraic geometry and Gröbner bases only deal with polynomial matrices. To be applicable for systems with general FIR filters, not just causal or anticausal filters, we need to extend the results from polynomial matrices to Laurent polynomial matrices. One method is to multiply both sides of (3.1) with a monomial of high enough degree. Thus $\boldsymbol{H}(\boldsymbol{z})$ is Laurent polynomial left invertible if and only if there exist an $P \times N$ polynomial matrix $\hat{\boldsymbol{G}}(\boldsymbol{z})$ such that

$$\hat{\boldsymbol{G}}(\boldsymbol{z})\boldsymbol{H}(\boldsymbol{z}) = \boldsymbol{z}^{\boldsymbol{k}}\boldsymbol{I}_P \tag{3.4}$$

for some integer vector $\boldsymbol{k}$. But finding a suitable integer vector $\boldsymbol{k}$ might require an extensive search. However, by generalizing Theorem 2 from [63], we have a simple algorithm to determine whether the given Laurent polynomial matrix is invertible or not.

Proposition 2. *Suppose $\boldsymbol{H}(\boldsymbol{z})$ is an $N \times P$ Laurent polynomial matrix. Consider the $(N+P) \times P$ matrix*

$$\boldsymbol{H}'(\boldsymbol{z}, w) = \begin{pmatrix} \boldsymbol{z}^{\boldsymbol{m}} \boldsymbol{H}(\boldsymbol{z}) \\ (1 - z_1 z_2 ... z_M w) \boldsymbol{I}_P \end{pmatrix} \tag{3.5}$$

where $\boldsymbol{m} \in \mathbb{N}^M$ is such that $\boldsymbol{z}^{\boldsymbol{m}} \boldsymbol{H}(\boldsymbol{z})$ is a polynomial matrix, w is a new variable, and $\boldsymbol{I}_P$ is a $P \times P$ identity matrix. Then $\boldsymbol{H}(\boldsymbol{z})$ is Laurent polynomial left invertible if and only if $\boldsymbol{H}'(\boldsymbol{z}, w)$ is a polynomial left invertible.

Proof. If $\boldsymbol{H}(\boldsymbol{z})$ is Laurent polynomial left invertible, then $\boldsymbol{z}^{\boldsymbol{m}} \boldsymbol{H}(\boldsymbol{z})$ is also Laurent polynomial left invertible. Then there exists a polynomial matrix $\boldsymbol{G}(\boldsymbol{z}) = (g_{ij}(\boldsymbol{z}))$ satisfying (3.4). Among these $\boldsymbol{k}$, pick one for which $\boldsymbol{m}' \in \mathbb{Z}_+^{\mathbb{M}}$ is the least integer vector. Let m_0 be the maximal entry of $\boldsymbol{m}' = \{m_1, ..., m_M\}$. If $m_0 = 0$, then $\boldsymbol{H}(\boldsymbol{z})$ is polynomial left invertible, so is $\boldsymbol{H}'(\boldsymbol{z}, w)$. Otherwise, m_0 is positive. Now let

$$g'_{ij}(\boldsymbol{z}, w) = \begin{cases} w^{m_0} \prod_{k=1}^{M} z_k^{m_0 - m_k} g_{ij}(\boldsymbol{z}), & i = 1, ..., P;\ j = 1, ..., N; \\ \sum_{k=0}^{m_0 - 1} (\prod_{l=1}^{M} z_l^k) w^k, & \text{if } i = j - N; \\ 0, & \text{otherwise.} \end{cases}$$

Let $\boldsymbol{G}'(\boldsymbol{z}, w) = (g'_{ij}(\boldsymbol{z}, w))$ be the corresponding $P \times (P+N)$ matrix. Then by a straightforward computation, we can conclude that $\boldsymbol{G}'(\boldsymbol{z}, w)$ is a polynomial left inverse of $\boldsymbol{H}'(\boldsymbol{z}, w)$.

Now suppose $\boldsymbol{H}'(\boldsymbol{z}, w)$ is polynomial left invertible. There exists $\boldsymbol{G}'(\boldsymbol{z}, w)$ such that $\boldsymbol{G}'(\boldsymbol{z}, w) \boldsymbol{H}'(\boldsymbol{z}, w) = \boldsymbol{I}$ with $\boldsymbol{G}'(\boldsymbol{z}, w) = \big(g'_{ij}(\boldsymbol{z}, w)\big)$. Set

$$\boldsymbol{G}(\boldsymbol{z}) = (\boldsymbol{z}^{-\boldsymbol{m}} g'_{ij}(\boldsymbol{z}, \prod_{k=1}^{M} z_k^{-1}))_{i=1,...,P;\ j=1,...,N}.$$

Then by a straightforward computation, we have $\boldsymbol{G}(\boldsymbol{z}) \boldsymbol{H}(\boldsymbol{z}) = \boldsymbol{I}$ and $\boldsymbol{G}(\boldsymbol{z})$ is a Laurent polynomial matrix. Hence $\boldsymbol{H}(\boldsymbol{z})$ is Laurent polynomial left invertible. □

Example 6. *Is $\boldsymbol{H}(\boldsymbol{z}) = \begin{pmatrix} z_1 & z_1 \\ z_2^2 + 3 & z_2^2 + 1 \end{pmatrix}$ invertible? Clearly it is not polynomial invertible because the determinant is zero when z_1 is zero. To verify that the matrix is Laurent polynomial left invertible, we need to introduce a new variable and the $\boldsymbol{H}'(\boldsymbol{z}, w)$ from (3.5) and test the invertibility of $\boldsymbol{H}'(\boldsymbol{z}, w)$.*

```
>ring R=0,(z(1),z(2),w),dp;

>matrix H'[4][2]=z(1),z(1),z(2)^2+3,z(2)^2+1,1-z(1)*z(2)*w,0,0,1-z(1)*z(2)*w;

>print(H');
```

```
z(1),z(1),
z(2)^2+3,z(2)^2+1,
-z(1)*z(2)*w+1,0,
0,-z(1)*z(2)*w+1
>module S=transpose(H');
>option(redSB);
>print(std(S));
1,0,
0,1
```

This implies that $\boldsymbol{H}(\boldsymbol{z})$ is Laurent polynomial left invertible.

3.2 Proposed Algorithms

3.2.1 Computation of Left Inverses

From Proposition 1 and Proposition 2, we introduce two new algorithms to generate an inverse matrix by using Gröbner bases if the given matrix is invertible.

Algorithm 3 (Particular Polynomial Inverse)**.** The computational algorithm for a polynomial left inverse matrix.

Input: $N \times P$ polynomial matrix $\boldsymbol{H}(\boldsymbol{z})$ over $\mathbb{C}[z_1, ..., z_M]$.

Output: $P \times N$ polynomial matrix $\boldsymbol{G}(\boldsymbol{z})$, if it exists.

1. Compute the reduced Gröbner basis of $\{\boldsymbol{h}_1(\boldsymbol{z}), ..., \boldsymbol{h}_N(\boldsymbol{z})\}$ where $\boldsymbol{h}_i(\boldsymbol{z})$ is a row of $\boldsymbol{H}(\boldsymbol{z})$ and the associated transformation matrix $\{W_{ij}(\boldsymbol{z})\}$ as defined in (3.2).

2. If the reduced Gröbner basis is $\{\boldsymbol{e}_i\}_{i=1,..,P}$, then output $(W_{ij}(\boldsymbol{z}))$. Otherwise, there is no solution.

Algorithm 4 (Particular Laurent Polynomial Inverse)**.** The computational algorithm for a Laurent polynomial left inverse matrix.

Input: $N \times P$ Laurent polynomial matrix $\boldsymbol{H}(\boldsymbol{z})$ with M variables.

Output: $P \times N$ Laurent polynomial matrix $\boldsymbol{G}(\boldsymbol{z})$, if it exists.

1. Multiply $\boldsymbol{H}(\boldsymbol{z})$ by a common monomial $\boldsymbol{z}^{\boldsymbol{m}}$ such that $\boldsymbol{H}'(\boldsymbol{z}, w)$ is polynomial matrix from Proposition 2.

2. Call Algorithm 3 with input $\boldsymbol{H}'(\boldsymbol{z}, w)$.

3. If the output of Algorithm 3 is $\boldsymbol{G}'(\boldsymbol{z}, w)$, then output $\boldsymbol{z}^{-\boldsymbol{m}}(\boldsymbol{G}'_{ij}(\boldsymbol{z}, \prod_{k=1}^{M} z_k^{-1}))_{i=1,...,P;\, j=1,...,N}$. Otherwise, there is no solution.

Example 7. *Find an inverse of* $\boldsymbol{H}(z_1, z_2) = \begin{pmatrix} 1 & 3z_2 \\ 2z_1+1 & 0 \\ 3 & z_1 \\ 3z_2 & 5 \end{pmatrix}$. *By Example 5, we know that* $\boldsymbol{H}(z_1, z_2)$ *is invertible.*

```
>matrix U[2][2]=unitmat(2);
```
% U is the 2×2 *identity matrix*

```
>matrix G[2][4]=transpose(lift(transpose(H),U));
```
% lift is function that returns a transformation matrix L where $U = H^T * L$

```
>print(G);
2/179z(1),18/179z(2)-1/179,-6/179z(2)+60/179,-12/179z(1),
12/179z(1),3/895z(2)-6/179,-36/179z(2)+2/179,-2/895z(1)+1/5
>print(G*H);
1,0,
0,1
```

Thus $\boldsymbol{G}(z_1, z_2)$ *is a left inverse of* $\boldsymbol{H}(z_1, z_2)$.

Example 8. *Find an inverse of* $\boldsymbol{H}(\boldsymbol{z}) = \begin{pmatrix} z_1 & z_1 \\ z_2^2+3 & z_2^2+1 \end{pmatrix}$. *By Example 6, we know that* $\boldsymbol{H}(\boldsymbol{z})$ *is Laurent polynomial left invertible. To calculate a left inverse using* **Singular**:

```
>matrix H'[4][2]=z(1),z(1),z(2)^2+3,z(2)^2+1,1-z(1)*z(2)*w,0,0,1-z(1)*z(2)*w;
>matrix U[2][2]=unitmat(2);
>matrix G'[2][4]=transpose(lift(transpose(H'),U));
>print(G');
-1/2*z(2)^3*w-1/2*z(2)*w,1/2*z(1)*z(2)*w,1,0,
1/2*z(2)^3*w+3/2*z(2)*w,-1/2*z(1)*z(2)*w,0,1
>print(G'*H');
1,0,
0,1
```

According to the above algorithm, $\boldsymbol{G}(\boldsymbol{z}) = \begin{pmatrix} -\frac{1}{2}z_1^{-1}z_2^2 - \frac{1}{2}z_1^{-1} & \frac{1}{2} \\ \frac{1}{2}z_1^{-1}z_2^2 + \frac{3}{2}z_1^{-1} & -\frac{1}{2} \end{pmatrix}$ *is a left inverse of* $\boldsymbol{H}(\boldsymbol{z})$.

Rajagopal and Potter explore the computation of the synthesis part of an M-variate perfect reconstruction FIR filter. Their algorithm [49, 48] first computes every maximal minor of $\boldsymbol{H}(\boldsymbol{z})$ and their corresponding adjoint matrices. Then it uses them to compute an inverse of $\boldsymbol{H}(\boldsymbol{z})$. The size of the set of maximal minors is $\binom{N}{P}$, which could be large if N and P were greatly different. When the difference of N and P is large, we find in practice that the algorithm is extremely slow. In order to avoid the problem that the computation of maximal minors poses, our Algorithm 4 computes an inverse directly by using the computation of the reduced Gröbner bases for modules. Park presents similar algorithms in [42] which use a different approach from the algorithm we propose. The only difference is the transformation function. Park's approach transforms the Laurent polynomials into polynomials by multiplying a series of elementary matrices while our approach simply transforms Laurent polynomials into polynomials by just multiplying by a large enough monomial. Therefore our approach is simpler and provides a closed form formula to compute an inverse.

When one designs a filter bank, one would like to estimate the degree of inverse matrices. Caniglia et al. [10] propose a upper bound on the degree of $N \times N$ invertible matrix $\boldsymbol{K}(\boldsymbol{z})$ such that $\boldsymbol{K}(\boldsymbol{z})\boldsymbol{H}(\boldsymbol{z}) = \begin{bmatrix} \boldsymbol{I}_P \\ \boldsymbol{0} \end{bmatrix}$ and the degree bound of $\deg(\boldsymbol{K}(\boldsymbol{z}))$ is optimal in order.

Proposition 3. *[10] Assume that $\boldsymbol{H}(\boldsymbol{z})$ is an $N \times P$ invertible matrix in M variables. Let $\deg(\boldsymbol{H}(\boldsymbol{z}))$ be the maximum of the degrees of the entries of $\boldsymbol{H}(\boldsymbol{z})$ and let $d = \deg(\boldsymbol{H}(\boldsymbol{z})) + 1$. Then there exists an $N \times N$ invertible matrix $\boldsymbol{K}(\boldsymbol{z})$ such that*

$$\boldsymbol{K}(\boldsymbol{z})\boldsymbol{H}(\boldsymbol{z}) = \begin{bmatrix} \boldsymbol{I}_P \\ \boldsymbol{0} \end{bmatrix}$$

and $\deg(\boldsymbol{K}(\boldsymbol{z}))$ is $(Pd)^{O(M)}$.

This suggests that the maximum degree of the entries of the $P \times N$ inverse matrix $\boldsymbol{G}(\boldsymbol{z})$ is also less than or equal to $(Pd)^{O(M)}$.

3.2.2 Characterization of Inverses

Algorithm 3 and Algorithm 4 do not guarantee that the inverse would be well behaved. In this section, we refer to some results that characterize the set of all inverses. Once we have a particular inverse, we can parametrize the set of all inverses.

Theorem 5 (Zhou)**.** *[62, 61] Suppose $\boldsymbol{H}(\boldsymbol{z})$ is an $N \times P$ polynomial matrix and $\tilde{\boldsymbol{G}}(\boldsymbol{z})$ is a $P \times N$ polynomial matrix such that $\tilde{\boldsymbol{G}}(\boldsymbol{z})\boldsymbol{H}(\boldsymbol{z}) = \boldsymbol{I}$. Then $\boldsymbol{G}(\boldsymbol{z})$ is an polynomial inverse matrix of $\boldsymbol{H}(\boldsymbol{z})$ if and only if $\boldsymbol{G}(\boldsymbol{z})$ can be written as*

$$\boldsymbol{G}(\boldsymbol{z}) = \tilde{\boldsymbol{G}}(\boldsymbol{z}) + \boldsymbol{A}(\boldsymbol{z})(\boldsymbol{I} - \boldsymbol{H}(\boldsymbol{z})\tilde{\boldsymbol{G}}(\boldsymbol{z})) \tag{3.6}$$

where $\boldsymbol{A}(\boldsymbol{z})$ is an arbitrary $P \times N$ polynomial matrix.

Theorem 6 (Park)**.** *[43] Suppose $\boldsymbol{H}(\boldsymbol{z})$ is an $N \times P$ polynomial matrix and $\tilde{\boldsymbol{G}}(\boldsymbol{z})$ is a $P \times N$ polynomial matrix such that $\tilde{\boldsymbol{G}}(\boldsymbol{z})\boldsymbol{H}(\boldsymbol{z}) = \boldsymbol{I}$. Let $\boldsymbol{h}_1, \boldsymbol{h}_2, ..., \boldsymbol{h}_N$ be row vectors of $\boldsymbol{H}(\boldsymbol{z})$. Then $\boldsymbol{G}(\boldsymbol{z})$ is an polynomial* *inverse matrix of* $\boldsymbol{H}(\boldsymbol{z})$ *if and only if* $\boldsymbol{G}(\boldsymbol{z})$ *can be written as*

$$\boldsymbol{G}(\boldsymbol{z}) = \tilde{\boldsymbol{G}}(\boldsymbol{z}) + \boldsymbol{A}(\boldsymbol{z})\mathrm{Syz}(\boldsymbol{h}_1, \boldsymbol{h}_2, ..., \boldsymbol{h}_N) \tag{3.7}$$

where $\boldsymbol{A}(\boldsymbol{z})$ *is an arbitrary polynomial matrix and* Syz *is the syzygy [1] of* $\{\boldsymbol{h}_1, \boldsymbol{h}_2, ..., \boldsymbol{h}_N\}$*.*

Remark 1. *Both of these theorems hold exactly the same where polynomial is replaced by Laurent polynomial.*

Zhou's method provides a simple characterization of inverses which is easy to compute while Park's method is more complicated to compute. However the matrix size of the free parameter $\boldsymbol{A}(\boldsymbol{z})$ in Theorem 5 is $P \times N$, while the smallest possible matrix size of $\boldsymbol{A}(\boldsymbol{z})$ in Theorem 6 is $P \times (N - P)$ in theory. Though syzygy provided by **Singular** does not necessary attain this optimal size, the used matrix size for $\boldsymbol{A}(\boldsymbol{z})$ in Park's method in general is smaller than Zhou's method.

Example 9. *Let be* $\boldsymbol{H}(\boldsymbol{z}) = \begin{pmatrix} z_1 & z_1+1 \\ z_2+z_1 & z_1 \\ 3 & z_1+2 \\ z_1 & z_2 \end{pmatrix}$*. Find the size of* $\boldsymbol{A}(z_1, z_2)$ *from Theorem 6 using* **Singular***.*

```
>ring R=0,(z(1),z(2)),dp;
>matrix H[4][2]=z(1),z(1)+1,z(2)+z(1),z(1),3,z(1)+2,z(1),z(2);
>option(redSB);
>matrix S=transpose(syz(transpose(H)));% where syz computes the syzygy
> print(S);
S[1,1],S[1,2],z(2)^2+z(1),z(1)-z(2)-3,
S[2,1],z(1)^2-z(1)-3,z(1)*z(2)+z(1)+z(2),0,
S[3,1],S[3,2],z(2)^3-z(1)-z(2),-z(2)^2-z(1)-4*z(2)
```

where $S[i,j]$ *is some long polynomial expression. Thus the required free parameter* $\boldsymbol{A}(z_1, z_2)$ *in Theorem 6 is a* 2×3 *matrix. It is not the optimal matrix size, namely* 2×2*. But the size of* $\boldsymbol{A}(z_1, z_2)$ *in Zhou's method is* 2×4*. Therefore applying Park's method using* **Singular** *would lead to smaller size of* $\boldsymbol{A}(\boldsymbol{z})$ *in this case.*

3.3 Conclusion

In this chapter we studied the inverse problem of a Laurent polynomial matrices. Such matrices arise in FIR filter banks as polyphase matrices. We present a condition for the left inverse matrix problem. Then we proposes algorithms to find a particular left inverse. Once having a particular solution, we can parametrize the set of all left inverses where an optimal solution can be found.

Chapter 4

Generic Invertibility

In Section 4.2, we prove that when $N - P \geq M$, then a polynomial matrix of degree at most k is generically polynomial (resp. : Laurent polynomial) left invertible; on the other hand, when $N - P < M$, then a polynomial matrix of degree at most k is generically polynomial (resp. : Laurent polynomial) noninvertible in Section 4.3. Based on this result, we give some applications and present a fast algorithm to find a particular inverse in Section 4.4.1.

4.1 Lebesgue Measure and Generic Property

When designing filter banks, an important question is how likely it is that the synthesis part of the perfect reconstruction filter banks exists. If it does not exist, then in general we are not able to reconstruct the original signal.

In [62], Zhou and Do made the following conjectures.

Conjecture 1. *Suppose $\boldsymbol{H}(\boldsymbol{z})$ is an $N \times P$ M-variate polynomial (resp. : Laurent polynomial) matrix with $N \geq P$. If $N - P \geq M$, then it is* "almost surely" *polynomial (resp. : Laurent polynomial) left invertible. Otherwise, it is* "almost surely" *polynomial (resp. : Laurent polynomial) left noninvertible.*

Rajagopal and Potter made another conjecture related to "almost surely" invertible in their paper [49].

Corollary 6 in [49]: Suppose $\boldsymbol{H}(\boldsymbol{z})$ is an $N \times P$ M-variate polynomial matrix with $N > P$. If $\binom{N}{P} > M$, then it is "*almost surely*" invertible.

Unfortunately, Corollary 6 in [49] is not correct. Please refer to Zhou's thesis [61] for more details.

Suppose the Conjecture 1 posed by Zhou and Do is true. If we design filter banks such that $N - P \geq M$, then "almost surely" there exists a synthesis part of the filter banks which is able to reconstruct the original signal perfectly.

However, Zhou and Do did not give a precise definition of "almost surely". In order to have the appropriate language, we employ the concept of Lebesgue measure and the concept of "hold generically".

Notation 1. *Let λ_n denote the* $2n$-dimensional Lebesgue measure.

In 2-dimensional plane, it is obvious that any "simple" line in plane has zero area. In 3-dimensional space, we also know that any "simple" surface has zero volume. To generalize this property, we have the following lemma.

Lemma 2. *[24, p.9] Let f be holomorphic (which means infinitely differentiable) in the domain $D \subset \mathbb{C}^n$, and suppose f is not identically zero. Then $\lambda_n(\{z \in D \mid f(z) = 0\}) = 0$.*

Definition 25 (Generic)**.** *[14] A property is said to* hold generically *for polynomials $f_1, .., f_n$ of degree at most $k_1, ..., k_n$ if there is a nonzero polynomial F in the coefficients of the f_i such that the property holds for $f_1, ..., f_n$ whenever the polynomial $F(f_1, ..., f_n)$ is nonvanishing.*

Intuitively, a property of polynomials is generic if it holds for "almost all" polynomials.

Example 10. *[14] The property "$f(x) = c_2x^2 + c_1x + c_0$ has two distinct solutions" is generic.*

Proof. Let F be a polynomial of the coefficients of $f = c_2x^2 + c_1x + c_0$ given by

$$F = c_2(c_1^2 - 4c_2c_0)$$

Suppose $F(f)$ is nonzero (i.e. $c_2(c_1^2 - 4c_2c_0) \neq 0$). Then $c_2 \neq 0$ and $c_1^2 - 4c_2c_0 \neq 0$. So f has two distinct solutions. Therefore by the above definition, $f(x) = c_2x^2 + c_1x + c_0$ has two distinct solutions generically. □

Lemma 3. *If a property of polynomials of degree at most $k_1, ..., k_n$ in m variables is generic, then the coefficient space $\mathcal{C}$ of polynomials whose polynomials failed to satisfy the property is measure zero and nowhere dense.*

Proof. By the definition of hold generically, there exists a nonzero polynomial F in the coefficients of the f_i such that the property fails to satisfy for $f_1, ..., f_n$ for which the polynomial $F(f_1, ..., f_n)$ is vanishing. Let R_i be the set of M-variate polynomials of degree less than or equal to k_i. By lemma 2,

$$\lambda_l(\{(f_1, ..., f_n) \in \prod_{i=1}^{n} R_i \mid F(f_1, ..., f_n) = 0\}) = 0$$

where $l = \binom{k_1+m}{m} + ... + \binom{k_n+m}{m}$ is the dimension of the coefficient space. Thus, the coefficient space $\mathcal{C}$ of polynomials whose polynomials failed to satisfy the property is measure zero. To show the set is nowhere dense, it is equivalent to show that the closure of the set contains no open set. Suppose it contains an open ball $B(\epsilon)$ with some radius $\epsilon > 0$. Since $F^{-1}(\{0\})$ is a closed set, $\overline{\mathcal{C}}$ is also in $F^{-1}(\{0\})$. Thus, $F^{-1}(\{0\})$ contains the open ball $B(\epsilon)$. However, this contradicts the fact that $F^{-1}(\{0\})$ is measure zero. Therefore, the coefficient space of polynomials whose polynomials failed to satisfy the property is nowhere dense. □

The immediate consequence is that if $f_1, ..., f_n$ are drawn independently from a probability distribution with respect to the Lebesgue measure, the property of $f_1, ..., f_n$ holds with probability one. Furthermore, suppose $\tilde{f}_0, ..., \tilde{f}_n$ satisfies the property. Since the coefficient space $\mathcal{C}$ of polynomials whose polynomials failed to satisfy the property is nowhere dense, there exists an open ball $B(\epsilon)$ around $\tilde{f}_0, ..., \tilde{f}_n$ for some $\epsilon > 0$ such that the property is satisfied within the open ball $B(\epsilon)$. This shows that the system with the property is robust [27].

4.2 Generically Invertible when $N - P \geq M$

To prove our main theorem in this section, we need to employ the resultant of the polynomials.

Theorem 7 (Resultant)**.** *[15, p.80] If we fix positive degrees $k_0, ..., k_n$, then there is a unique nonzero polynomial called the* resultant $\mathrm{RES}_{(k_0,...,k_n)} \in \mathbb{C}[\bigcup_{i=1}^{n} \{u_{i,j}\}_{j=1,...,\binom{k_i+n}{n}}]$ *where the variables $u_{i,j}$ correspond to the coefficients of i-th polynomial. Then we have the following property:*
If $F_0, ..., F_n \in \mathbb{C}[x_0, ..., x_n]$ are homogeneous of degrees $k_0, ..., k_n$, then $F_0, ..., F_n$ have a nontrivial common zero over $\mathbb{C}$ if and only if $\mathrm{RES}_{(k_0,...,k_n)}(F_0, ..., F_n) = 0$.

Remark 2. *Let $\boldsymbol{H}(\boldsymbol{z})$ be an $N \times P$ Laurent polynomial matrix in M variables. Then $\boldsymbol{H}(\boldsymbol{z})$ is Laurent polynomial left invertible if and only if there exist a monomial $\boldsymbol{z}^{\boldsymbol{l}}$ and a polynomial matrix $\boldsymbol{H}'(\boldsymbol{z})$ which is Laurent polynomial left invertible such that*

$$\boldsymbol{H}(\boldsymbol{z}) = \boldsymbol{z}^{-\boldsymbol{l}} \boldsymbol{H}'(\boldsymbol{z}).$$

Remark 3. *Let $\boldsymbol{H}(\boldsymbol{z})$ be a polynomial matrix. If $\boldsymbol{H}(\boldsymbol{z})$ is polynomial left invertible, then $\boldsymbol{H}(\boldsymbol{z})$ is Laurent polynomial left invertible. But the converse is not true in general. Also if $\boldsymbol{H}(\boldsymbol{z})$ is Laurent polynomial left noninvertible, then $\boldsymbol{H}(\boldsymbol{z})$ is polynomial left noninvertible. But the converse is not true in general also.*

Example 11. *Let (z) be a 1×1 matrix. It is not polynomial left invertible matrix but it is a Laurent polynomial left invertible matrix as $(z^{-1})(z) = 1$.*

By Remark 2, it is enough to consider the polynomial matrices.

Now we can translate the first half of Conjecture 1 into the following mathematical frameworks:

Theorem 8. *If $N - P \geq M$ and $k > 0$, then an $N \times P$ polynomial M-variate matrix $\boldsymbol{H}(\boldsymbol{z})$ of degree at most k is generically polynomial left invertible.*

Proof. The strategy of this proof is to find a nonzero polynomial F such that $F(\boldsymbol{H}(\boldsymbol{z})) = 0$ for every noninvertible matrix $\boldsymbol{H}(\boldsymbol{z})$ of degree at most k.

Let $\boldsymbol{Z} = (z_0, ..., z_M)$. If $f(\boldsymbol{z}) = f_0(\boldsymbol{z}) + f_1(\boldsymbol{z}) + ... + f_l(\boldsymbol{z})$ is the decomposition of the polynomial $f(\boldsymbol{z})$ into sums of forms $f_i(\boldsymbol{z})$ of degree i, then the homogenization $\overline{f}(\boldsymbol{Z})$ of $f(\boldsymbol{z})$ of degree k is defined to be $\overline{f}(\boldsymbol{Z}) = z_0^k f_0(\boldsymbol{z}) + z_0^{k-1} f_1(\boldsymbol{z}) + ... + z_0^{k-l} f_l(\boldsymbol{z})$. Let $\boldsymbol{h}_i(\boldsymbol{Z})$ be the ith row of an $N \times P$ matrix $\overline{\boldsymbol{H}}(\boldsymbol{z})$. Let $t_i(\boldsymbol{Z})$ be the determinant of the $P \times P$ submatrix containing $\boldsymbol{h}_i(\boldsymbol{Z}), \boldsymbol{h}_{i+1}(\boldsymbol{Z}), ..., \boldsymbol{h}_{i+P-1}(\boldsymbol{Z})$. Define ϕ to be a function such that

$$\boldsymbol{H}(\boldsymbol{z}) \mapsto (t_1(\boldsymbol{Z}), t_2(\boldsymbol{Z}), ..., t_{M+1}(\boldsymbol{Z}))^T.$$

Rajagopal and Potter in [49, 48] show that if $\boldsymbol{H}(\boldsymbol{z})$ is noninvertible and $N \geq P$, then the $P \times P$ maximal minors of $\boldsymbol{H}(\boldsymbol{z})$ have a common zero. Suppose $(\tilde{z}_1/\tilde{z}_0, \tilde{z}_2/\tilde{z}_0, ..., \tilde{z}_M/\tilde{z}_0)$ is a solution of the maximal minors of $\boldsymbol{H}(\boldsymbol{z})$ where $\tilde{z}_0 \neq 0$. Then $(\tilde{z}_0, \tilde{z}_1, \tilde{z}_2, ..., \tilde{z}_M)$ is a nonzero solution of maximal minors of $\overline{\boldsymbol{H}}(\boldsymbol{Z})$. Since $\{t_1, ..., t_{M+1}\}$ is a part of the subset of the set of maximal minors of $\overline{\boldsymbol{H}}(\boldsymbol{Z})$, this implies that $\phi(\boldsymbol{H}(\boldsymbol{z}))$ have a nontrivial common zero. Therefore, by the property of the resultant shown in Theorem 7, we know

$$\mathrm{RES}_{(Pk,...,Pk)}(\phi(\boldsymbol{H}(\boldsymbol{z}))) = 0 \tag{4.1}$$

for all noninvertible matrices $\boldsymbol{H}(\boldsymbol{z})$ of degree at most k. The $\mathrm{RES}_{(Pk,...,Pk)}$ and t_i are polynomials, so is $\mathrm{RES}_{(Pk,...,Pk)} \circ \phi$. Last but not least, we need to show $\mathrm{RES}_{(Pk,...,Pk)} \circ \phi$ is not a zero function. Let

$$\boldsymbol{T}(\boldsymbol{z}) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ z_1^k & 1 & \dots & 0 \\ z_2^k & z_1^k & \ddots & 0 \\ \vdots & \vdots & \ddots & 1 \\ z_M^k & z_{M-1}^k & \ddots & z_1^k \\ 0 & z_M^k & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & z_M^k \\ 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{pmatrix}$$

be an $N \times P$ matrix. Suppose $\mathrm{RES}_{(Pk,...,Pk)}(\phi(\boldsymbol{T}(\boldsymbol{z}))) = 0$. By Theorem 7, we know that t_i's have a nontrivial common zero. i.e. there exists $\tilde{\boldsymbol{Z}}$ a nonzero solution such that

$$t_{M+1}(\tilde{\boldsymbol{Z}}) = \tilde{z}_M^{Pk} = 0.$$

This implies $\tilde{z}_M = 0$. If $\tilde{z}_M = 0$, then $t_M(\tilde{z}_0, \tilde{z}_1, ..., \tilde{z}_{M-1}, 0) = \tilde{z}_{M-1}^{Pk} = 0$. Thus $\tilde{z}_{M-1} = 0$. Continuing the process, we can conclude $\tilde{z}_0 = \tilde{z}_1 = ... = \tilde{z}_M = 0$. This contradicts the assumption that $\tilde{\boldsymbol{Z}}$ is nontrivial. So $\mathrm{RES}_{(Pk,...,Pk)}(\phi(\boldsymbol{T}(\boldsymbol{z}))) \neq 0$. Therefore $\mathrm{RES}_{(Pk,...,Pk)} \circ \phi$ is not zero function. By the definition of hold generically, we conclude that $\boldsymbol{H}(\boldsymbol{z})$ of degree at most k is generically polynomial left invertible matrix. □

Theorem 9. *If $N - P \geq M$ and $k > 0$, then an $N \times P$ polynomial M-variate matrix $\boldsymbol{H}(\boldsymbol{z})$ of degree at most k is generically Laurent polynomial left invertible.*

Proof. By above remark, we know that if a polynomial matrix $\boldsymbol{H}(\boldsymbol{z})$ is Laurent polynomial left noninvertible, then $\boldsymbol{H}(\boldsymbol{z})$ is also polynomial left noninvertible. According to Theorem 8, this shows that $\mathrm{RES}_{(Pk,...,Pk)} \circ \phi(\boldsymbol{H}(\boldsymbol{z})) = 0$ for all Laurent polynomial left noninvertible polynomial matrix $\boldsymbol{H}(\boldsymbol{z})$. □

4.3 Generically Noninvertible when $N - P < M$

Projective n-space $\mathbb{P}^n$ is the set of equivalence classes of $(n+1)$-tuples $(a_0, ..., a_n)$ of elements of $\mathbb{C}$, not all zero, under the equivalence relation given by $(a_0, ..., a_n) \sim (\lambda a_0, ..., \lambda a_n)$ for all nonzero $\lambda \in \mathbb{C}$.

The following lemma depends heavily on commutative ring theory and algebraic geometry. For detail definition of ring, ideal, radical ideal, and prime ideal, please refer to [40] and [28]. For the propose of our proof, we only need the following definition.

Definition 26 (Height)**.** *The* height *of a prime ideal* $\mathrm{ht}\, p$ *is the supremum of the lengths* n *of strictly descending chains* $p = p_0 \supset p_1 \supset ... \supset p_n$ *of prime ideals. For an arbitrary ideal* I, $\mathrm{ht}\, I = \inf\{\mathrm{ht}\, p \mid I \subset p,\ p \textit{ is prime ideal}\}$.

Lemma 4. *Given $\boldsymbol{H}(\boldsymbol{z})$ is $N \times P$ polynomial matrix in M variables of degree at most $k > 0$ and $N \geq P$. Let*

$$V(\{m_i\}) := \{\boldsymbol{Z} \in \mathbb{P}^n \mid m_i(\boldsymbol{Z}) = 0 \quad \textit{for all } i = 1, ..., \tbinom{N}{P}\}$$

where m_i is a maximal minor of $\overline{\boldsymbol{H}}(\boldsymbol{Z})$ with some ordering and $\overline{\boldsymbol{H}}(\boldsymbol{Z})$ is the homogenization of $\boldsymbol{H}(\boldsymbol{z})$ of degree k. Then $V(\{m_i\})$ is empty if and only if $\mathrm{ht}\,\langle m_i\rangle = M + 1$. Therefore if $V(\{m_i\})$ is empty, then $N - P \geq M$. In other words, if $N - P < M$, then $V(\{m_i\})$ is nonempty.

Proof. Since m_i is homogeneous, then the unit does not lie in $\langle m_i \rangle$. This implies that $\langle m_i \rangle \neq \mathbb{C}[x_0, ..., x_n]$. By [14, p.370] and the definition of radical ideal, $V(\{m_i\})$ is empty if and only if $\langle \sqrt{m_i} \rangle = \langle x_0, ..., x_M \rangle$. It is easy to see that $\text{ht}\, \langle \sqrt{m_i} \rangle = M + 1$. Since $\text{ht}\, \langle m_i \rangle = \text{ht}\, \langle \sqrt{m_i} \rangle$, the height of $\langle m_i \rangle$ is also $M + 1$. Macaulay in [39, p.54] proved that $\text{ht}\, \langle m_i \rangle \leq N - P + 1$. Therefore if $V(\{m_i\})$ is empty, then $N - P \geq M$. In other word, if $N - P < M$, then $V(\{m_i\})$ is nonempty. ☐

Definition 27 (Weak-Zero)**.** *[61] A point in $\mathbb{P}^n$ is said to be* weak-zero *if at least one of its coordinates is zero.*

Lemma 5. *The polynomial matrix $\boldsymbol{H}(\boldsymbol{z})$ is Laurent polynomial invertible if and only if the set $V(\{m_i\})$ contains only weak-zeros where $\boldsymbol{H}(\boldsymbol{z})$, V and m_i are same as above lemma.*

Proof. Follows immediately by Proposition 5.2 in [22]. ☐

Now we can prove the second half of Conjecture 1.

Theorem 10. *If $N - P < M$ and $k > 0$, then an $N \times P$ polynomial M-variate matrix $\boldsymbol{H}(\boldsymbol{z})$ of degree at most k is generically Laurent polynomial left noninvertible.*

Proof. The strategy of the proof is the same as above Theorem 8. We will find a nonzero polynomial F such that $F(\boldsymbol{H}(\boldsymbol{z})) = 0$ for every Laurent polynomial left invertible polynomial matrix $\boldsymbol{H}(\boldsymbol{z})$.

If $N < P$, then every polynomial matrix is left noninvertible. Now consider $\boldsymbol{H}(\boldsymbol{z})$ is invertible. Let c_{ij} be a coefficient for the constsnt term of $h_{ij}(\boldsymbol{z})$ where $\boldsymbol{H}(\boldsymbol{z}) = (h_{ij}(\boldsymbol{z}))$. Define a function F_1 such that

$$\boldsymbol{H}(\boldsymbol{z}) \mapsto \prod_{\substack{i=1,...,N \\ j=1,...,P}} c_{ij}. \tag{4.2}$$

If $h_{ij}(z_1, ..., z_{N-P+1}, 0, ..., 0) = 0$ for some i, j, then it implies $c_{ij} = 0$. This shows that $F(\boldsymbol{H}(\boldsymbol{z})) = 0$ in (4.4). If $h_{ij}(z_1, ..., z_{N-P+1}, 0, ..., 0) \neq 0$ for all i, j, then $\boldsymbol{H}(z_1, ..., z_{N-P+1}, 0, ..., 0)$ is also invertible because there exists Laurent polynomial matrix $\boldsymbol{G}(\boldsymbol{z})$ such that $\boldsymbol{G}(\boldsymbol{z})\boldsymbol{H}(\boldsymbol{z}) = \boldsymbol{I}$ and $\boldsymbol{G}(z_1, ..., z_{N-P+1}, 0, ..., 0)$ is well-defined. We can now assume that $M = N - P + 1$. Define $t_i(\boldsymbol{Z})$ to be the same as Theorem 8. Let $t_j^{(i)} = t_j(z_0, ..., \overset{i\text{-th}}{0}, ..., z_M)$. Define θ_i to be a function such that

$$\boldsymbol{H}(\boldsymbol{z}) \mapsto (t_1^{(i)}, ..., t_M^{(i)})^T$$

for $i = 0, ..., M$. By Lemma 4 and Lemma 5 and the fact that $\{t_1^{(i)}(\boldsymbol{Z}), ..., t_M^{(i)}(\boldsymbol{Z})\}$ is the subset of the set of maximal minors of $\overline{\boldsymbol{H}}(\boldsymbol{Z})$, it implies that $\theta_i(\boldsymbol{H}(\boldsymbol{z}))$ have a nonzero common zero for some $i = 0, ..., M$. By the property of the resultant shown in Theorem 7, we know that given any Laurent polynomial left invertible polynomial matrix $\boldsymbol{H}(\boldsymbol{z})$,

$$\mathrm{RES}_{(Pk,\ldots,Pk)}(\theta_i(\boldsymbol{H}(\boldsymbol{z}))) = 0 \quad \text{for some } i = 0, ..., M. \tag{4.3}$$

The $\mathrm{RES}_{(Pk,\ldots,Pk)}$ and $t_j^{(i)}$ are polynomials, so is $\mathrm{RES}_{(Pk,\ldots,Pk)} \circ \theta_i$. Lastly, we need to show $\mathrm{RES}_{(Pk,\ldots,Pk)} \circ \theta_i$ is not a zero function. Let

$$\boldsymbol{T}_i(\boldsymbol{z}) = \begin{pmatrix} z_1^k & 0 & \ldots & 0 \\ z_2^k & z_1^k & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ z_{i-1}^k & z_{i-2}^k & \ddots & z_1^k \\ 1 & z_{i-1}^k & \ddots & \vdots \\ z_{i+1}^k & 1 & \ddots & z_{i-1}^k \\ \vdots & \vdots & \ddots & 1 \\ z_M^k & z_{M-1}^k & \ddots & z_{i+1}^k \\ 0 & z_M^k & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & z_M^k \end{pmatrix}$$

be an $N \times P$ matrix. Suppose $\mathrm{RES}_{(Pk,\ldots,Pk)}(\theta_i(\boldsymbol{T}_i(\boldsymbol{z}))) = 0$. By Theorem 7, we know that $\{t_1^{(i)}, ..., t_M^{(i)}\}$ have a nontrivial common zero. i.e. there exists $\tilde{\boldsymbol{Z}}$ a nonzero solution such that

$$t_M^{(i)}(\tilde{z}_0, ..., \overset{i\text{-th}}{0}, ..., \tilde{z}_M) = \tilde{z}_M^{Pk} = 0.$$

This implies $\tilde{z}_M = 0$. If $\tilde{z}_M = 0$, then

$$t_{M-1}^{(i)}(\tilde{z}_0, ..., \overset{i\text{-th}}{0}, ..., \tilde{z}_{M-1}, 0) = \tilde{z}_{M-1}^{Pk} = 0.$$

Thus $\tilde{z}_{M-1} = 0$. Continuing the process, we can conclude $\tilde{z}_0 = \tilde{z}_1 = ... = \tilde{z}_M = 0$. This contradicts the assumption that $\tilde{\boldsymbol{Z}}$ is nontrivial. So $\mathrm{RES}_{(Pk,\ldots,Pk)}(\theta_i(\boldsymbol{T}_i(\boldsymbol{z}))) \neq 0$. Therefore $\mathrm{RES}_{(Pk,\ldots,Pk)} \circ \theta_i$ is not zero function. Now let

$$F = F_1 \times \prod_{i=0}^{M} \mathrm{RES}_{(Pk,\ldots,Pk)} \circ \theta_i. \tag{4.4}$$

By (4.3) and (4.2), $F(\boldsymbol{H}(\boldsymbol{z})) = 0$ for all Laurent polynomial left invertible polynomial matrix $\boldsymbol{H}(\boldsymbol{z})$. This

	$N-P \geq M$	$N-P < M$
$\boldsymbol{H}(\boldsymbol{z})$	generic invertible	generic noninvertible

Table 4.1: Polynomial (resp. : Laurent polynomial) invertibility of $\boldsymbol{H}(\boldsymbol{z})$

			N			
			1	2	3	4
M=1	P	1	0	500	500	500
		2	0	0	500	500
		3	0	0	0	500
		4	0	0	0	0
M=2	P	1	0	0	500	500
		2	0	0	0	500
		3	0	0	0	0
		4	0	0	0	0
M=3	P	1	0	0	0	500
		2	0	0	0	0
		3	0	0	0	0
		4	0	0	0	0

Table 4.2: Inversibility test for a random polynomial matrix generator with different N, P and M in 500 test cases

shows that if $N - P < M$, then a polynomial matrix $\boldsymbol{H}(\boldsymbol{z})$ of degree at most k is generically Laurent polynomial left invertible. □

Theorem 11. *If $N - P < M$ and $k > 0$, then an $N \times P$ polynomial M-variate matrix $\boldsymbol{H}(\boldsymbol{z})$ of degree at most k is generically polynomial left noninvertible.*

Proof. By Remark 3, we know that if a polynomial matrix $\boldsymbol{H}(\boldsymbol{z})$ is polynomial left invertible, then $\boldsymbol{H}(\boldsymbol{z})$ is also Laurent polynomial left invertible. According to Theorem 10, this shows that $F(\boldsymbol{H}(\boldsymbol{z})) = 0$ for all polynomial left invertible polynomial matrix $\boldsymbol{H}(\boldsymbol{z})$. □

4.4 Simulation and Applications on Generic Invertibility

From Table 4.2, we used a random polynomial matrix generator to generate polynomial matrices with each entry of degree less than or equal to 4 and the random coefficients are from 1 to 100. In each value of N, P and M, we ran 500 samples to test the inversibility. We found out that they agreed with our theorems.

These theorems lead to some applications. For image deconvolution from multiple FIR blur filters, Harikumar and Bresler in [27, 26] show that perfect reconstruction is almost surely, when there are at least three channels. Since image is two dimension (i.e. $M = 2$) and the downsampling rate is just one (i.e.

$P = 1$), by Theorem 9, we know that the perfect reconstruction is almost surely if number of channels is greater than two (i.e. $N \geq 3$). Therefore Harikumar and Bresler's image deconvolution is a special case of our main theorem.

Another application is that we can have an alternative approach in designing multidimensional filter banks. We can freely design the analysis side first such that it satisfies the condition (i.e. $N - P \geq M$). Then , by Theorem 9 and Lemma 3, we can almost surely find a perfect reconstruction inverse for the synthesis polyphase matrix.

4.4.1 Fast Computation of Left Inverses

By Theorem 9, we know we should design the filter banks such that $N - P \geq M$. Suppose $N - P \geq M$. Since $\boldsymbol{H}(\boldsymbol{z})$ is a Laurent polynomial matrix, there exists $\boldsymbol{l} \in \mathbb{N}^M$ such that $\boldsymbol{z}^{\boldsymbol{l}}\boldsymbol{H}(\boldsymbol{z})$ is a polynomial matrix and is generically Laurent polynomial left invertible. However, at the same time, the $\boldsymbol{z}^{\boldsymbol{l}}\boldsymbol{H}(\boldsymbol{z})$ is generically polynomial left invertible by Theorem 8. Due to this fact, we can improve our Algorithm 4.

Algorithm 5 (Faster Version)**.** The computational algorithm for a Laurent polynomial left inverse matrix.
Input: $N \times P$ Laurent polynomial matrix $\boldsymbol{H}(\boldsymbol{z})$ with M variables.
Output: $P \times N$ Laurent polynomial matrix $\boldsymbol{G}(\boldsymbol{z})$, if it exists.
1. Multiply $\boldsymbol{H}(\boldsymbol{z})$ by a common monomial $\boldsymbol{z}^{\boldsymbol{l}}$ such that $\boldsymbol{z}^{\boldsymbol{l}}\boldsymbol{H}(\boldsymbol{z})$ are polynomial matrix.
2. Call Algorithm 3 with the input $\boldsymbol{z}^{\boldsymbol{l}}\boldsymbol{H}(\boldsymbol{z})$.
3. If the output of Algorithm 3 is $\boldsymbol{J}(\boldsymbol{z})$, then output $\boldsymbol{z}^{-\boldsymbol{l}}\boldsymbol{J}(\boldsymbol{z})$.
4. Otherwise call Algorithm 4.

Since Algorithm 3 does not need to introduce any new variable and the matrix is smaller, the computation of Algorithm 3 is faster than Algorithm 4. Moreover, as we mentioned before $\boldsymbol{z}^{\boldsymbol{l}}\boldsymbol{H}(\boldsymbol{z})$ is generically polynomial left invertible, so most of the time we would perform Algorithm 3 in step 3, which leads to less frequent calling of Algorithm 4 in step 4. Therefore, Algorithm 5 is faster than Algorithm 4 in most cases.

Example 12. *Compare the processing time between Algorithm 4 and Algorithm 5. Let* $\boldsymbol{H}(z_1, z_2)$

$$= \begin{pmatrix} 4z_1 & 7{z_1}^{-1}z_2^2 + 2 + 10{z_1}^{-1} \\ 1 + 10{z_1}^{-1} & 10z_1 + 3z_2 \\ 7z_1 + 9z_2 + 10{z_1}^{-1}z_2 + 10{z_1}^{-1} & 0 \\ 8{z_1}^{-1}z_2^2 + 10 + 4{z_1}^{-1} & 6{z_1}^{-1}z_2^2 \end{pmatrix}$$

be a Laurent polynomial matrix. Then let $\boldsymbol{H}'(z_1, z_2, w)$

$$= \begin{pmatrix} 4z_1^2 & 7z_2^2 + 2z_1 + 10 \\ z_1 + 10 & 10z_1^2 + 3z_1z_2 \\ 7z_1^2 + 9z_1z_2 + 10z_2 + 10 & 0 \\ 8z_2^2 + 10z_1 + 4 & 6z_2^2 \\ 1 - z_1z_2w & 0 \\ 0 & 1 - z_1z_2w \end{pmatrix}$$

be a polynomial matrix according to Proposition 2.

To calculate a Laurent polynomial left inverse using Algorithm 4:

`>system(‘‘--min-time", ‘‘0.02");`

`>timer=1;` *% The time of each command is printed*

`>int t=timer;` *% initialize t by timer*

`>matrix U[2][2]=unitmat(2);`

`>matrix G’[2][6]=transpose(lift(transpose(H’),U));`

`//used time:  0.23 sec.` *% using a desktop PC*

Then the left inverse is $({z_1}^{-1}\boldsymbol{G}'(z_1, z_2, z_1^{-1}z_2^{-1}))_{i=1,2,j=1,..,4}$.

To calculate a Laurent polynomial left inverse using Algorithm 5 in **Singular***:*

`>matrix U[2][2]=unitmat(2);`

`>matrix J[2][4]=transpose(lift(transpose(z(1)*H),U));`

`//used time:  0.06 sec.`

Then the left inverse is ${z_1}^{-1}\boldsymbol{J}(z_1, z_2)$.

This agrees that Algorithm 5 is faster than Algorithm 4.

4.4.2 Inverse with Perturbation

Suppose $N - P \geq M$. Let $\boldsymbol{H}(\boldsymbol{z})$ be an $N \times P$ noninvertible matrix in M variables. Since the set of noninvertible matrices of degree at most k is measure zero and nowhere dense when $N - P \geq M$ by Lemma 3. Therefore , if $\boldsymbol{E}(\boldsymbol{z})$ is some small perturbation, then the $(\boldsymbol{H} + \boldsymbol{E})(\boldsymbol{z})$ is generically invertible. i.e. There exists $\boldsymbol{G}^{(\boldsymbol{E})}(\boldsymbol{z})$ such that

$$\boldsymbol{G}^{(\boldsymbol{E})}(\boldsymbol{z})(\boldsymbol{H} + \boldsymbol{E})(\boldsymbol{z}) = \boldsymbol{I}$$

Thus $\boldsymbol{G}^{(\boldsymbol{E})}(\boldsymbol{z})\boldsymbol{H}(\boldsymbol{z}) = \boldsymbol{I} - \boldsymbol{G}^{(\boldsymbol{E})}(\boldsymbol{z})\boldsymbol{E}(\boldsymbol{z})$. Let $\boldsymbol{E}_{ij}(\boldsymbol{z})$ be an $N \times P$ matrix which differs from the zero matrix by having ϵ in the ij-component instead of 0. Suppose $(\boldsymbol{H} + \boldsymbol{E}_{ij})(\boldsymbol{z})$ is invertible for some i, j. Define $||.||_C$ be a sum of magnitude square of coefficients. Then

$$\begin{aligned}
&||\boldsymbol{G}^{(\boldsymbol{E}_{ij})}(\boldsymbol{z})\boldsymbol{H}(\boldsymbol{z}) - \boldsymbol{I}||_C \\
=&||\boldsymbol{G}^{(\boldsymbol{E}_{ij})}(\boldsymbol{z})\boldsymbol{E}_{ij}(\boldsymbol{z})||_C \\
=&\left|\left|\begin{pmatrix} 0 & \dots & 0 & \overbrace{\epsilon g_{1i}}^{j-\text{th}} & 0 & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & \epsilon g_{Pi} & 0 & \dots & 0 \end{pmatrix}\right|\right|_C \\
\leq&\epsilon^2||G_i^{(\boldsymbol{E}_{ij})}(\boldsymbol{z})||_F^2
\end{aligned}$$

where $||.||_F$ stands for the Frobenius norm and $G_i^{(\boldsymbol{E}_{ij})}(\boldsymbol{z})$ is synthesis filter which corresponds to $\boldsymbol{G}^{(\boldsymbol{E}_{ij})}(\boldsymbol{z})$. So if $\epsilon^2||G_i^{(\boldsymbol{E}_{ij})}(\boldsymbol{z})||_F^2$ is small, then it implies that $\boldsymbol{G}^{(\boldsymbol{E}_{ij})}(\boldsymbol{z})\boldsymbol{E}_{ij}(\boldsymbol{z}) \approx \boldsymbol{0}$. Thus $\boldsymbol{G}^{(\boldsymbol{E}_{ij})}(\boldsymbol{z})\boldsymbol{H}(\boldsymbol{z}) \approx \boldsymbol{I}$.

Example 13. *Consider the analysis filters*

$$\begin{aligned}
H_1(\boldsymbol{z}) &= (1 + z_1)(1 + z_2), \quad & H_2(\boldsymbol{z}) &= (1 - z_1)(1 - z_1 z_2), \\
H_3(\boldsymbol{z}) &= (1 - z_1)(z_1 - z_2), \quad & H_4(\boldsymbol{z}) &= (1 - z_2)(1 - z_1 z_2),
\end{aligned}$$

and the sampling matrix

$$\boldsymbol{D} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

By polyphase representation, we obtains an 4×2 *matrix* $\boldsymbol{H}(\boldsymbol{z})$*, which is noninvertible. However,* $(\boldsymbol{H}+\boldsymbol{E}_{11})(\boldsymbol{z})$ *is invertible for* $\epsilon = 1$, 0.1, 0.01, 0.001, 0.0001, *and* 0.00001*. We use the Algorithm 10 to find an optimal inverse given* $\boldsymbol{A}(\boldsymbol{z}) = (a_{ij})_{i=1,..,2;\, j=1,...,4}$*. In Table 4.3, we obverse that the* MSE *and* $\epsilon^2||G_1^{(\boldsymbol{E}_{11})}(\boldsymbol{z})||_F^2$ *both converge to* 31.6432 *and* 0.4892 *respectively when* ϵ *is getting smaller. In Figure 4.1, we obverse that the quality of the image is getting better if* ϵ *is getting smaller. However, the quality of images are almost the same if* $\epsilon \leq 0.1$*. To reduce the reconstruction errors due to additive white Gaussian noise, we need to minimum* $\sum_{i=1}^4 ||G_i^{(\boldsymbol{E}_{11})}(\boldsymbol{z})||_F^2$ *by Proposition 8. Therefore it would be a reasonable choice if we set* $\epsilon = 0.1$*, because* $\sum_{i=1}^4 ||G_i^{(\boldsymbol{E}_{11})}(\boldsymbol{z})||_F^2$*,* $\epsilon^2||G_1^{(\boldsymbol{E}_{11})}(\boldsymbol{z})||_F^2$*, and* MSE *are all relatively low.*

Question 1. *Suppose* $\boldsymbol{H}(\boldsymbol{z})$ *is closed to singular. Is* $\min_{\boldsymbol{G}\in\mathcal{G}} \sum_{i=1}^4 ||G_i(\boldsymbol{z})||_F^2$ *large? or suppose* $\min_{\boldsymbol{G}\in\mathcal{G}} \sum_{i=1}^4 ||G_i(\boldsymbol{z})||_F^2$ *large, does it mean* $\boldsymbol{H}(\boldsymbol{z})$ *is closed to singular?*

ϵ	1	0.1	0.01	0.001	0.0001	0.00001
MSE	775.677	43.267	31.821	31.651	31.6438	31.6432
$\epsilon^2 \|\|G_1^{(\boldsymbol{E}_{11})}(\boldsymbol{z})\|\|_F^2$	0.699	0.498	0.4899	0.4893	0.4892	0.4892
$\sqrt{\sum_{i=1}^4 \|\|G_i^{(\boldsymbol{E}_{11})}(\boldsymbol{z})\|\|_F^2}$	1.317	7.569	74.100	740.025	7399.341	73992.501

Table 4.3: Compute the MSE, $\epsilon^2 ||G_1^{(\boldsymbol{E}_{11})}(\boldsymbol{z})||_F^2$, and $\sqrt{\sum_{i=1}^4 ||G_i^{(\boldsymbol{E}_{11})}(\boldsymbol{z})||_F^2}$ by finding optimal inverse of $(\boldsymbol{H} + \boldsymbol{E}_{11})(\boldsymbol{z})$.

4.4.3 Left Invertible Matrices Completion

By Theorem 9 and Theorem 10, and $M \geq 1$, then an $N \times N$ matrix $\boldsymbol{H}(\boldsymbol{z})$ is generic noninvertible. Therefore, critically sampled filter banks are almost surely not perfect reconstruction.

Let $\boldsymbol{H}(\boldsymbol{z})$ be an $N \times P$ invertible matrix with $N \geq P$. Can we complete $\boldsymbol{H}(\boldsymbol{z})$ to be a square $N \times N$ invertible matrix $\overline{\boldsymbol{H}}(\boldsymbol{z})$ by adding $N - P$ columns to the matrix $\boldsymbol{H}(\boldsymbol{z})$? This question is one version of Serre's conjecture raised by Jean P. Serre in 1955 [52]. This problem of polynomial version was solved in 1976 by Quillen [47] and Suslin [54] independently. The problem of Laurent polynomial version was solved by Swan [55].

Proposition 4. *[47, 54, 55] Every left invertible $N \times P$ matrix (resp. : right invertible $P \times N$ matrix) with $N \geq P$ can be completed to a square invertible $N \times N$ matrix $\overline{\boldsymbol{H}}(\boldsymbol{z})$ by adding $N - P$ columns (resp. : rows) to the matrix $\boldsymbol{H}(\boldsymbol{z})$.*

Constructive proofs and algorithms are given by [38, 36, 35, 60, 46]. Heuristic algorithm is given by Park [42], which is extremely simple but it may not always work.

Since an $N \times N$ matrix $\boldsymbol{H}(\boldsymbol{z})$ is generic noninvertible for $M \geq 1$, it is very difficult to find an invertible square matrix. However, if $\boldsymbol{H}(\boldsymbol{z})$ is an $N \times (N - M)$ matrix in M variables, then $\boldsymbol{H}(\boldsymbol{z})$ is generic invertible by Theorem 9. Then by Proposition 4, the completion $\overline{\boldsymbol{H}}(\boldsymbol{z})$ of $\boldsymbol{H}(\boldsymbol{z})$ is invertible square matrix. Therefore, we can design critically sampled perfect reconstruction filter banks, but this kind of design cannot have a control on the analysis filters.If we want to have some control on the analysis filters, an alternative design is needed.

Theorem 12 (Generic Right Invertible version of Theorem 9)**.** *If $P - N \geq M$ and $k > 0$, then an $N \times P$ matrix $\boldsymbol{H}(\boldsymbol{z})$ of degree at most k is generically right invertible.*

Fact 1. *If an $N \times P$ right invertible matrix $\boldsymbol{H}(\boldsymbol{z})$, then the right invertible matrix of the completion $\overline{\boldsymbol{H}}(\boldsymbol{z})$ of $\boldsymbol{H}(\boldsymbol{z})$ is left invertible.*

By Theorem 12 and Fact 1, we can design the part of the analysis filters. First choose analysis filters

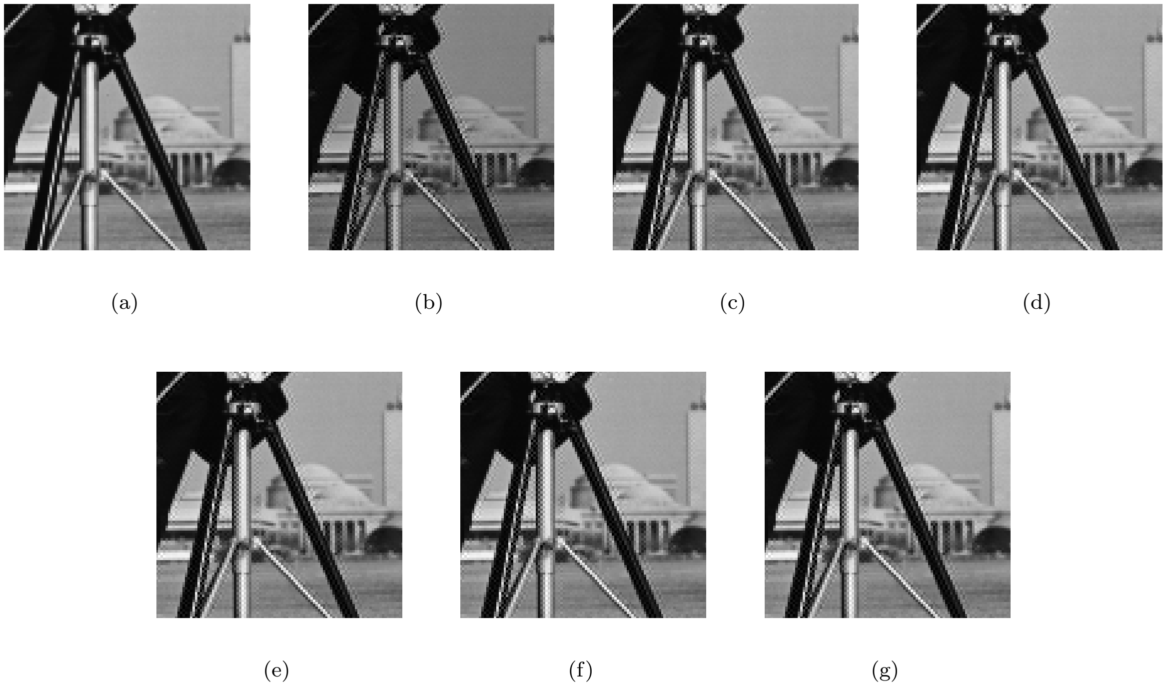


Figure 4.1: The reconstruction of the filter bank system $\mathcal{S}_{H_i,G_i^{(E_{11})},\boldsymbol{D}}$: (a) Original image. (b) Reconstruction for $\epsilon = 1$. (c) Reconstruction for $\epsilon = 0.1$. (d) Reconstruction for $\epsilon = 0.01$. (e) Reconstruction for $\epsilon = 0.001$. (f) Reconstruction for $\epsilon = 0.0001$. (g) Reconstruction for $\epsilon = 0.00001$.

$H_i(\boldsymbol{z})$ for $i = 1, ..., N-M$. Then by the polyphase representation with some sampling matrix $\boldsymbol{D}$ such that the sampling rate N, we have an $(N-M) \times N$ matrix $\boldsymbol{H}(\boldsymbol{z})$. By Theorem 12, the $\boldsymbol{H}(\boldsymbol{z})$ is generically right invertible. By Fact 1, there exists an $\boldsymbol{G}(\boldsymbol{z})$ such that $\boldsymbol{G}(\boldsymbol{z})\overline{\boldsymbol{H}}(\boldsymbol{z}) = I$. By (5.5) and (5.8), we obtain the analysis filters $\overline{H}_i(\boldsymbol{z})$ and the synthesis filters $G_i(\boldsymbol{z})$. Moreover, the filter bank system $\mathcal{S}_{\overline{H}_i,G_i,\boldsymbol{D}}$ is critically sampled perfect reconstruction while $\overline{H}_i(\boldsymbol{z}) = H_i(\boldsymbol{z})$ for $i = 1, ..., N-M$.

4.4.4 n-Parallel Filter Banks

To achieve a critical sampling, we introduce n-parallel filter banks. Now consider $N = nP$ and $N - P \geq M$. By Theorem 9 and the condition that $N - P \geq M$, we know filter bank systems are almost surely perfect reconstruction. We can design analysis filters $H_i(\boldsymbol{z})$ for $i = 1, ..., N$ such that perfect reconstruction with some synthesis filters $G_i(\boldsymbol{z})$ and some sampling matrix. But it is oversampled system. However, we can add some particular analysis filters and synthesis filters in parallel to accomplish a critical sampling. By the polyphase representation, we have an $N \times P$ matrix $\boldsymbol{H}_0(\boldsymbol{z})$. Then we can find a left inverse $\boldsymbol{G}_0(\boldsymbol{z})$ that

$$\boldsymbol{G}_0(\boldsymbol{z})\boldsymbol{H}_0(\boldsymbol{z}) = \boldsymbol{I}.$$

Now we use Proposition 4 to complete $\boldsymbol{H}_0(\boldsymbol{z})$ to be an $N \times N$ matrix

$$\overline{\boldsymbol{H}}_0(\boldsymbol{z}) = \left[\ \boldsymbol{H}_0(\boldsymbol{z}) \vdots \boldsymbol{H}_1(\boldsymbol{z}) \vdots \cdots \vdots \boldsymbol{H}_{n-1}(\boldsymbol{z})\ \right]$$

where $\boldsymbol{H}_i(\boldsymbol{z})$ is an $N \times P$ matrix for $i = 0, ..., n-1$. Since the completion $\overline{\boldsymbol{H}}_0(\boldsymbol{z})$ of $\boldsymbol{H}_0(\boldsymbol{z})$ is invertible, there exists an $N \times N$ matrix

$$\boldsymbol{G}(\boldsymbol{z}) = \begin{bmatrix} \boldsymbol{G}_0(\boldsymbol{z}) \\ \hdashline \boldsymbol{G}_1(\boldsymbol{z}) \\ \hdashline \vdots \\ \hdashline \boldsymbol{G}_{n-1}(\boldsymbol{z}) \end{bmatrix}$$

such that $\boldsymbol{G}(\boldsymbol{z})\overline{\boldsymbol{H}}_0(\boldsymbol{z}) = \boldsymbol{I}$. Thus, we have

$$\boldsymbol{G}_i(\boldsymbol{z})\boldsymbol{H}_j(\boldsymbol{z}) = \begin{cases} \boldsymbol{I} & i = j; \\ \boldsymbol{0} & i \neq j. \end{cases}$$

By the relationship between filters and polyphase matrices (see Section 5.2), we have the following:

$$H_i^{(k)}(\boldsymbol{z}) = \sum_{[\boldsymbol{l}_j] \in \mathcal{N}(\boldsymbol{D})} \boldsymbol{z}^{-\boldsymbol{l}_j^*} \{\boldsymbol{H}_k(\boldsymbol{z})\}_{ij}(\boldsymbol{z}^{\boldsymbol{D}}),$$

$$G_i^{(k)}(\boldsymbol{z}) = \sum_{[\boldsymbol{l}_j] \in \mathcal{N}(\boldsymbol{D})} \boldsymbol{z}^{\boldsymbol{l}_j^*} \{\boldsymbol{G}_k(\boldsymbol{z})\}_{ji}(\boldsymbol{z}^{\boldsymbol{D}})$$

for $i = 1, ..., N$ and $k = 0, ..., n-1$. Then the filter bank system $\mathcal{S}_{H_i^{(k)}, G_i^{(k)}, \boldsymbol{D}}$ is perfect reconstruction for $k = 0, ..., n-1$; whereas the filter bank system $\mathcal{S}_{H_i^{(k)}, G_i^{(l)}, \boldsymbol{D}}$ has zero output for $k \neq l$.

Example 14. *Consider $N = 4$ and $P = 2$. We choose a random analysis filters*

$$H_1(\boldsymbol{z}) = z_2^3 + 2z_2^2 + z_1 + z_2 + 2,$$

$$H_2(\boldsymbol{z}) = z_2^3 + z_1 z_2 + 2z_1 + z_2 + 2,$$

$$H_3(\boldsymbol{z}) = 2z_2^3 + 2z_2^2 + 2z_1 + 2z_2 + 1,$$

$$H_4(\boldsymbol{z}) = 2z_1 z_2 + z_2^2 + 2z_1 + z_2 + 1.$$

with a sampling matrix $\boldsymbol{D} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Let input X and Y be the images of the cameramen and Lena. By algorithm given by Park [42], we can find analysis filters $H_i^{(j)}$ and synthesis filters $G_i^{(j)}$ for $i = 1, ..., 4$ and $j = 1, 2$. In Fig. 4.3, we obverse that the image of the output W_i are the mixture of X and Y.

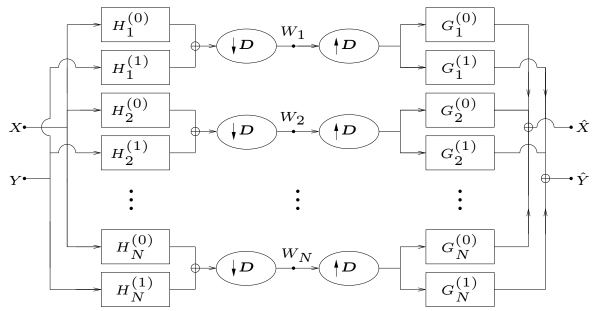


Figure 4.2: 2-Parallel Filter Banks.

4.5 Conclusion

We shows that there is a sharp phase transition on the invertibility depending on the size and dimension of a given Laurent polynomial matrix. Specifically when $N - P \geq M$, the $N \times P$ polynomial (resp. : Laurent polynomial) of M-variate matrix is generically invertible; whereas when $N - P < M$, the matrix is generically noninvertible. Using this sharp phase transition property, we give some applications and develop a fast algorithm to compute a particular left inverse for a given Laurent polynomial matrix.

These results suggest an alternative approach in designing multidimensional filter banks by freely generating filters for the analysis side first. If we allow an amount of oversampling (i.e. $N - P \geq M$), then we can almost surely find a perfect reconstruction inverse for the synthesis polyphase matrix. These results also have potential applications in multidimensional signal reconstruction from multichannel filtering and sampling.

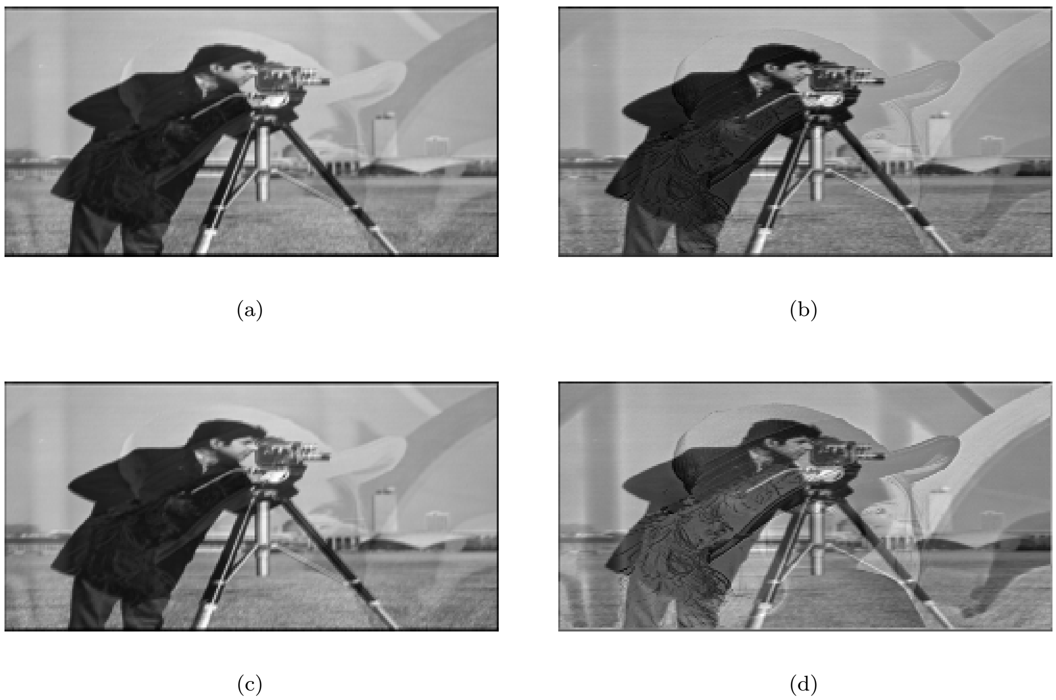


Figure 4.3: The output of analysis part of the filter bank system in Fig. 4.2: (a) the output W_1. (b) the output W_2. (c) the output W_3 . (d) the output W_4.

Chapter 5

Multidimensional Filter Bank Signal Reconstruction From Multichannel Acquisition

5.1 Introduction

In recent decades, multirate systems have played an increasingly important role in several engineering areas. The ideas of multirate systems have been extended from one dimensional systems to multidimensional systems; see [57, 59]. For multichannel acquisition application, the most common employed multirate systems are N-channel perfect reconstruction (PR) finite impulse response (FIR) uniform filter banks [8, 58, 20, 32].

In the traditional setting, the analysis filters and the sampling matrix are given (or estimated) in the uniform filter bank system. The goal is to find synthesis filters such that the system remains a perfect reconstruction for all input signals [12, 26, 44, 49, 63]. In this chapter, we relax the restriction that only the analysis filters are given. The new goal is to find a suitable sampling matrix and synthesis polyphase matrix which satisfy a perfect reconstruction condition. Suppose that we have an N-channel convolution system in M dimensions. Instead of taking all the data and applying multichannel deconvolution, we can first reduce the collected data set by an integer $M \times M$ sampling matrix $\boldsymbol{D}$ and still perfectly reconstruct the signal with a PR synthesis polyphase matrix (see Fig. 5.1). Of course, we want the sampling factor to be as large as possible because it would give us a minimum collected data set. To address this situation, we want to answer the following questions:

Problem 1. *Given a set of analysis filters, can we have a PR system with some sampling matrix $\boldsymbol{D}$ and FIR synthesis polyphase matrix?*

Problem 2. *If so, find a sampling matrix $\boldsymbol{D}$ having a maximal sampling factor and a particular PR FIR synthesis polyphase matrix?*

Problem 3. *Among all maximum density sampling matrices and all PR FIR synthesis polyphase matrices, can we find an optimal solution which provides a robust reconstruction in presence of noise?*

These questions can be approached differently in one dimensional (1D) and multidimensional (MD) cases. In 1D, since $\boldsymbol{D}$ is a scalar, we just need to do a linear search on sampling factors. In MD, however, then

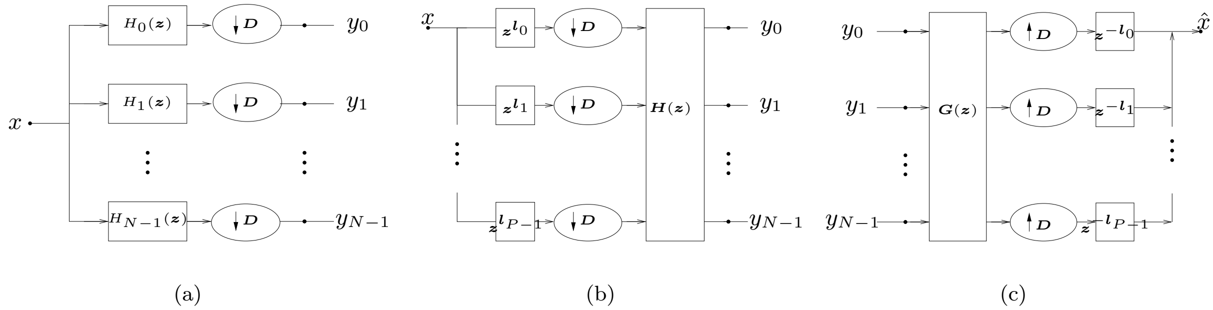


Figure 5.1: (a) Signal acquisition or analysis part: Multichannel convolution followed downsampling by $\boldsymbol{D}$. (b) Polyphase representation of the analysis part. (c) Synthesis polyphase reconstruction.

there are an infinite many matrices with a given sampling factor. We will address these two problems using Hermite normal form and Smith normal form. Although the system may not have a PR FIR synthesis polyphase matrix for given analysis filters and a given downsampling, it is possible that there exist some delays before the downsampling to provide the system to have a PR FIR synthesis matrix. We will develop an algorithm to find such delays if they exist.

The theory of frames is an essential tool to apply on oversampled filter banks when collected data are corrupted by noise [21, 18, 29, 17, 5]. One of applications is to study and estimate an optimal upper frame bound [6, 16]. We will apply the frame analysis to find an optimal PR FIR synthesis matrix with respect to different norms.

The outline of the chapter is the following. Section 5.2 states the general problem and has a review on polyphase representation. Section 5.3 provides necessary condition to have PR system with some sampling matrices and some FIR synthesis polyphase matrices, develops algorithms to find a sampling matrix with maximum sampling rate and FIR synthesis polyphase matrix for given FIR analysis filters, and has a brief discussion on the multichannel acquisition where delays before the downsampling are allowed. Section 5.4, finally, discusses the optimizations of synthesis FIR polyphase matrices with respect to different norms.

5.2 Problem Formulation

Let M, N and $\boldsymbol{D}$ be the dimension of signals, the number of channels and an $M \times M$ sampling matrix with integer entries respectively. Let P be the sampling factor at each channel, $P := |\det \boldsymbol{D}|$. Let $\mathcal{N}(\boldsymbol{D})$ be the quotient group of $\boldsymbol{D}\mathbb{Z}^M$ in $\mathbb{Z}^M$ (i.e. $\mathbb{Z}^M/\boldsymbol{D}\mathbb{Z}^M$). Without loss of generality, we make use of Vaidyanathan's definition [56] and denote $[\boldsymbol{l}_j]$ to be $\boldsymbol{l}_j + \boldsymbol{D}\mathbb{Z}^M \in \mathcal{N}(\boldsymbol{D})$ where $\boldsymbol{l}_j \in \{\boldsymbol{D}\boldsymbol{t} \mid \boldsymbol{t} \in [0,1)^M\}$. The definition of downsampling by $\boldsymbol{D}$ is defined as

$$x_d[\boldsymbol{n}] = x[\boldsymbol{D}\boldsymbol{n}].$$

The definition of upsampling by $\boldsymbol{D}$ is defined as

$$x_u[\boldsymbol{n}] = \begin{cases} x[\boldsymbol{D}^{-1}\boldsymbol{n}], & \text{if } \boldsymbol{n} \in \text{LAT}(\boldsymbol{D}); \\ 0, & \text{otherwise.} \end{cases}$$

where $\text{LAT}(\boldsymbol{D})$ denotes the set of all vectors of the form $\boldsymbol{D}\boldsymbol{m}$, $\boldsymbol{m} \in \mathbb{Z}^M$. Furthermore, $\text{LAT}(\boldsymbol{D})$ is an abelian group with ordinary addition. By the definition of group isomorphism, $\text{LAT}(\boldsymbol{D}) \cong \boldsymbol{D}\mathbb{Z}^M \subset \mathbb{Z}^M$.

Example 15. *The separable sampling lattice in two-dimensional with sampling rate* 2 *in both dimensions is an sampling matrix* $\boldsymbol{D} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. *Then*

$$\mathcal{N}(\boldsymbol{D}) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \Big/ \boldsymbol{D}\mathbb{Z}^M.$$

By the polyphase decomposition (PD) [57, 59] (a brief review is provided in the following section), the analysis and synthesis parts can be represented by an $N \times P$ matrix $\boldsymbol{H}(\boldsymbol{z})$ and $P \times N$ matrix $\boldsymbol{G}(\boldsymbol{z})$ shown in Fig. 5.1(b) and Fig. 5.1(c), respectively, where every element is a Laurent polynomial in $\boldsymbol{z}$. The PR condition $\hat{X}(\boldsymbol{z}) = X(\boldsymbol{z})$ is equivalent to $\boldsymbol{G}(\boldsymbol{z})\boldsymbol{H}(\boldsymbol{z}) = \boldsymbol{I}_P$.

The data acquisition can be modeled shown in Fig. 5.1(a). We consider $H_i(\boldsymbol{z})$ fixed (i.e. point spread function of sampling devices), but the sampling matrix $\boldsymbol{D}$ can be changed. We provide an efficient algorithm to find $\boldsymbol{D}$ having maximal sampling rate and FIR synthesis polyphase matrix $\boldsymbol{G}(\boldsymbol{z})$ such that PR condition hold. This is a generalization of multichannel deconvolution problem. (i.e. $\boldsymbol{D} = I$)

Let $\mathcal{N}(\boldsymbol{D}) = \{[\boldsymbol{l}_0], ..., [\boldsymbol{l}_{P-1}]\}$. By the polyphase representation, the polyphase discrete signal matrix is given by

$$\boldsymbol{X}(\boldsymbol{z}) = \begin{pmatrix} X_0(\boldsymbol{z}) \\ \vdots \\ X_{(P-1)}(\boldsymbol{z}) \end{pmatrix} \tag{5.1}$$

where

$$X_j(\boldsymbol{z}) = \sum_{\boldsymbol{m} \in \mathbb{Z}^M} x[\boldsymbol{D}\boldsymbol{m} + \boldsymbol{l}_j]\boldsymbol{z}^{-\boldsymbol{m}} \tag{5.2}$$

and

$$X(\boldsymbol{z}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^M} x[\boldsymbol{n}]\boldsymbol{z}^{-\boldsymbol{n}} = \sum_{[\boldsymbol{l}_j]\in\mathcal{N}(\boldsymbol{D})} X_j(\boldsymbol{z}^{\boldsymbol{D}})\boldsymbol{z}^{-\boldsymbol{l}_j}.$$

By the polyphase representation, the polyphase analysis matrix $\boldsymbol{H}(\boldsymbol{z})$ is defined as $N \times P$ matrix

$$\boldsymbol{H}(\boldsymbol{z}) = \begin{pmatrix} H_{0,0}(\boldsymbol{z}) & \cdots & H_{0,P-1}(\boldsymbol{z}) \\ \vdots & \cdots & \vdots \\ H_{N-1,0}(\boldsymbol{z}) & \cdots & H_{N-1,P-1}(\boldsymbol{z}) \end{pmatrix} \tag{5.3}$$

where

$$H_{i,j}(\boldsymbol{z}) = \sum_{\boldsymbol{m}\in\mathbb{Z}^M} h_i[\boldsymbol{D}\boldsymbol{m} - \boldsymbol{l}_j]\boldsymbol{z}^{-\boldsymbol{m}}. \tag{5.4}$$

Then the analysis filters are given by

$$H_i(\boldsymbol{z}) = \sum_{[\boldsymbol{l}_j]\in\mathcal{N}(\boldsymbol{D})} \boldsymbol{z}^{\boldsymbol{l}_j} H_{i,j}(\boldsymbol{z}^{\boldsymbol{D}}). \tag{5.5}$$

Similarly by the polyphase representation, the polyphase synthesis matrix $\boldsymbol{G}(\boldsymbol{z})$ is defined as $P \times N$ matrix

$$\boldsymbol{G}(\boldsymbol{z}) = \begin{pmatrix} G_{0,0}(\boldsymbol{z}) & \cdots & G_{0,N-1}(\boldsymbol{z}) \\ \vdots & \cdots & \vdots \\ G_{P-1,0}(\boldsymbol{z}) & \cdots & G_{P-1,N-1}(\boldsymbol{z}) \end{pmatrix} \tag{5.6}$$

where

$$G_{i,j}(\boldsymbol{z}) = \sum_{\boldsymbol{m}\in\mathbb{Z}^M} g_i[\boldsymbol{D}\boldsymbol{m} + \boldsymbol{l}_j]\boldsymbol{z}^{-\boldsymbol{m}}. \tag{5.7}$$

Then the synthesis filters are given by

$$G_i(\boldsymbol{z}) = \sum_{[\boldsymbol{l}_j]\in\mathcal{N}(\boldsymbol{D})} \boldsymbol{z}^{-\boldsymbol{l}_j} G_{i,j}(\boldsymbol{z}^{\boldsymbol{D}}). \tag{5.8}$$

We transform (5.14) into z-transform domain, this gives us that

$$\boldsymbol{Y}(\boldsymbol{z}) = \boldsymbol{H}(\boldsymbol{z})\boldsymbol{X}(\boldsymbol{z})$$

where $\boldsymbol{Y}(\boldsymbol{z}) = (Y_0(\boldsymbol{z}), ..., Y_{N-1}(\boldsymbol{z}))^T$ with $Y_k(\boldsymbol{z}) = \sum_{\boldsymbol{m}\in\mathbb{Z}^M} y_k[\boldsymbol{m}]z^{-\boldsymbol{m}}$. Similarly, we transform (5.15) into

z-transform domain, this given us that

$$\hat{\boldsymbol{X}}(\boldsymbol{z}) = \boldsymbol{G}(\boldsymbol{z})\boldsymbol{Y}(\boldsymbol{z})$$

where $\hat{\boldsymbol{X}}(\boldsymbol{z}) = (\hat{X}_0(\boldsymbol{z}), ..., \hat{X}_{P-1}(\boldsymbol{z}))^T$ with $\hat{X}_i(\boldsymbol{z}) = \sum_{\boldsymbol{m}\in\mathbb{Z}^M} \hat{x}[\boldsymbol{m}\boldsymbol{D} + \boldsymbol{l}_i]z^{-\boldsymbol{m}}$

Theorem 13. *The $\hat{x}[\boldsymbol{n}] = x[\boldsymbol{n}]$ if and only if $\boldsymbol{G}(\boldsymbol{z})\boldsymbol{H}(\boldsymbol{z}) = \boldsymbol{I}$.*

Proof. Followed by the above discussion. □

Algorithm 6 (Analysis Polyphase Matrix Computation)**.** *Input: $\{H_0(\boldsymbol{z}), ..., H_{N-1}(\boldsymbol{z})\}$, $\mathcal{N}(\boldsymbol{D}) := \{[\boldsymbol{l}_0], ..., [\boldsymbol{l}_{P-1}]\}$, and $\boldsymbol{D}$.*

Output: $N \times P$ polyphase matrix $\boldsymbol{H}(\boldsymbol{z})$.

Set $H_{i,j}(\boldsymbol{z}) = 0$ for all $i = 0, ..., N-1$, $j = 0, ..., P-1$;

For i from 0 to $N-1$

  For each $\boldsymbol{n}$ in the support of $H_i(\boldsymbol{z})$

  (with $h_{i,\boldsymbol{n}}$ as the corresponding filter coefficient)

    For j from 0 to $P-1$

      $\boldsymbol{w} = D^{-1}(\boldsymbol{n} - \boldsymbol{l}_j)$;

      If w is an integer vector,

        then $H_{i,j}(\boldsymbol{z}) = H_{i,j}(\boldsymbol{z}) + h_{i,\boldsymbol{n}}\boldsymbol{z}^{\boldsymbol{w}}$;

    End for loop;

  End for loop;

End for loop;

Output $\boldsymbol{H}(\boldsymbol{z})$;

5.3 PR Synthesis Polyphase Matrix Algorithm

5.3.1 Representation of Sampling Matrices

As we mentioned in the introduction, when the dimension M of signals is greater than one, there are infinite many sampling matrices with a same sampling rate. In this subsection, we employ Hermite normal form to show that there are only finite number of sampling matrices for a given sampling rate up to equivalence classes. Next, applying Smith norm form, we develop an algorithm to find a sampling matrix and synthesis polyphase matrix satisfying the PR condition.

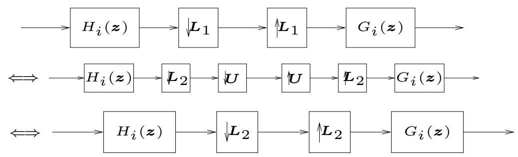


Figure 5.2: Equivalent filters with sampling matrices $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$

Proposition 5. *[33] $LAT(\boldsymbol{L}_1) = LAT(\boldsymbol{L}_2)$ if and only if $\boldsymbol{L}_1 = \boldsymbol{L}_2\boldsymbol{U}$, where $\boldsymbol{U}$ is unimodular integer matrix (i.e. $|\det \boldsymbol{U}| = 1$).*

By above proposition and Fig. 5.2, this shows that the system with sampling matrix $\boldsymbol{L}_1$ is a PR if and only if the system with sampling matrix $\boldsymbol{L}_2$ is a PR. Though this proposition greatly reduced the search space, we still do not know whether the search space is finite or not. Moreover, it does not provide us any search method. We will address these by using Hermite Normal Form.

Theorem 14 (Hermite Normal Form)**.** *[41] Given an $M \times M$ nonsingular integer-valued matrix $\boldsymbol{D}$, there exists an $M \times M$ unimodular matrix $\boldsymbol{K}$ such that $\boldsymbol{DK} = \boldsymbol{H}$, the Hermite normal form of $\boldsymbol{D}$ whose entries satisfy*

$$
\begin{aligned}
&h_{i,j} = 0, \quad \forall j > i, \\
&h_{i,i} > 0, \quad \forall i, \\
&h_{i,j} \leq 0 \text{ and } |h_{i,j}| < h_{i,i}, \quad \forall j < i.
\end{aligned}
$$

Furthermore, the Hermite normal form for $\boldsymbol{D}$ is unique.

This theorem implies that the set of nonsingular integer matrices of a given absolute determinant forms a equivalence relationship. We define $C(P)$ to be a set of Hermite normal forms for a given absolute determinant P. Now we want to know what the size of $C(P)$ is.

Theorem 15. *[41] Let $|C_M(P)|$ denote be the size of $C_M(P)$, then*

$$
\begin{aligned}
|C_1(P)| &= 1, \\
|C_M(P)| &= \sum_{q|P} q|C_{M-1}(q)|, \quad M \geq 2.
\end{aligned}
$$

When $M = 2$, then we have $|C_2(P)| = \sum_{q|P} q$. Robin [50] proves that $|C_2(P)| = \mathrm{O}(P \log\log P)$.

Example 16. *If $M = 2$ and $P = 4$, then $|C_2(4)| = \sum_{q|4} q = 1 + 2 + 4 = 7$. The complete set $C_2(4)$ of* *representative sampling matrices with sampling rate* 4 *is:*

$$\begin{pmatrix} 1 & 0 \\ -3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix},$$
$$\begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \text{ and } \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}.$$

For the following algorithms, we will employ the Smith normal form. For a detail discussion, please refer to [41].

Theorem 16 (Smith Normal Form)**.** *[41] Any nonsingular integer matrix* $\boldsymbol{D}$ *can always be decomposed as* $\boldsymbol{D} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{V}$ *where* $\boldsymbol{U}$ *and* $\boldsymbol{V}$ *are unimodular integer matrices and* $\boldsymbol{\Lambda}$ *is diagonal matrix with positive-integer elements such that the diagonal* λ_i*'s of* $\boldsymbol{\Lambda}$ *satisfy that* λ_i *is a divisor of* λ_{i+1}*.* $\boldsymbol{\Lambda}$ *is unique for a given* $\boldsymbol{M}$*.*

There are many algorithms for efficiently computing the Smith normal form. Among all algorithms, The Storijohann's algorithm using modular techniques to compute Smith normal form and transformation matrices gives the best known complexity analysis for the integer matrices [53].

Corollary 1. *Let* $\boldsymbol{D} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{V}$ *be a Smith normal form. Define* $\Phi(\boldsymbol{l}) = \boldsymbol{U}\boldsymbol{l}$*. Then* $\mathcal{N}(\boldsymbol{D}) = \Phi(\mathcal{N}(\boldsymbol{\Lambda}))$*.*

Proof. Follows immediately by Proposition 5 and the fact that Φ is a group isomorphism. □

Example 17. *In this example, we would like to find* $\mathcal{N}(\boldsymbol{D})$ *for* $\boldsymbol{D} = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$*. By Smith normal form decomposition, the sampling matrix* $\boldsymbol{D} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{V}$ *where* $\boldsymbol{U} = \begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix}$*,* $\boldsymbol{\Lambda} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$*, and* $\boldsymbol{V} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$*. It is easy to see that* $\mathcal{N}(\boldsymbol{\Lambda}) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\} \Big/ \boldsymbol{\Lambda}\mathbb{Z}^M$*. Then* $\mathcal{N}(\boldsymbol{D}) = \Phi(\mathcal{N}(\boldsymbol{\Lambda})) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\} \Big/ \boldsymbol{D}\mathbb{Z}^M$*.*

Algorithm 7. The algorithm of finding a sampling matrix and synthesis filters satisfying the PR condition for the input sampling rate.

Input: FIR analysis filters $H_i(\boldsymbol{z})$ *and a sampling rate* P*.*

Output: sampling matrix $\boldsymbol{D}$ *and FIR synthesis polyphase matrix together with the set* $\mathcal{N}(\boldsymbol{D})$*, if it exists.*

1 Set $C(P)$ *be a set of the Hermite normal forms for* P*.*

2. Take $\boldsymbol{D}$ *from* $C(P)$ *and set* $C(P) := C(P) - \{\boldsymbol{D}\}$*.*

3. Let $\boldsymbol{D} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{V}$ *be a Smith normal form. Find* $\mathcal{N}(\boldsymbol{D})$ *by Corollary 1.*

4. Call Algorithm 6 to compute the analysis polyphase matrix $\boldsymbol{H}(\boldsymbol{z})$*.*

5. Call Algorithm 4; Output the FIR synthesis polyphase matrix $\boldsymbol{G}(\boldsymbol{z})$ together with the set $\mathcal{N}(\boldsymbol{D})$ and the sampling matrix $\boldsymbol{D}$ if the inverse exist.

6. If $C(P)$ is not empty, then goto 2. Otherwise, there is no solution.

5.3.2 Existence of PR Synthesis Polyphase Matrices

Instead of starting for running every sampling matrix in $C(P)$ and looking for a PR synthesis polyphase matrix, it would be more efficient if we can find a necessary condition on analysis filters to have a PR system with some sampling matrix and some synthesis polyphase matrix.

Proposition 6. *The polynomial analysis filters $H_i(\boldsymbol{z})$ have no common solution if and only if the system provide a PR for some sampling matrix and some polynomial synthesis polyphase matrix.*

Proof. Suppose polynomial analysis filters $H_0(\boldsymbol{z}), H_1(\boldsymbol{z}), ..., H_{N-1}(\boldsymbol{z})$ have no common zero. To show the system provide a PR for some sampling matrix and some polynomial synthesis polyphase matrix, it is enough to show when sampling matrix $D = I$ (i.e. no downsampling and no downsampling within the filter bank), there exists some polynomial synthesis filters to have a PR. Since the variety

$$V(\{H_0(\boldsymbol{z}), H_1(\boldsymbol{z}), ..., H_{N-1}(\boldsymbol{z})\}) = \emptyset$$

Then by Weak Nullstellensatz Theorem [14], the submodule $\langle H_0(\boldsymbol{z}), H_1(\boldsymbol{z}), ..., H_{N-1}(\boldsymbol{z})\rangle$ of $\mathbb{C}[\boldsymbol{z}]$ contains an unit. Therefore there exists polynomials $G_i(\boldsymbol{z})$ such that

$$\sum_{i=0}^{N-1} G_i(\boldsymbol{z})H_i(\boldsymbol{z}) = 1.$$

On the other hand, we want to show if polynomial analysis filters $H_0(\boldsymbol{z}), H_1(\boldsymbol{z}), ..., H_{N-1}(\boldsymbol{z})$ have common zero, then the system does not provide a PR for all sampling matrices and all polynomial synthesis polyphase matrices. Suppose there exists $\boldsymbol{D}$ and $\{G_i(\boldsymbol{z})\}$ satisfying the PR condition.
By polyphase decomposition, we have

$$\boldsymbol{G}(\boldsymbol{z})\boldsymbol{H}(\boldsymbol{z}) = \boldsymbol{I}.$$

This is always true that

$$\boldsymbol{G}(\boldsymbol{z}^{\boldsymbol{D}})\boldsymbol{H}(\boldsymbol{z}^{\boldsymbol{D}}) = \boldsymbol{I}. \tag{5.9}$$

Let $\boldsymbol{r}(\boldsymbol{z})$ be the first row of $\boldsymbol{G}(\boldsymbol{z}^{\boldsymbol{D}})$ and $\boldsymbol{c}_j(\boldsymbol{z}^{\boldsymbol{D}})$ be the j-th column of $\boldsymbol{H}(\boldsymbol{z}^{\boldsymbol{D}})$. Let

$$\boldsymbol{v}(\boldsymbol{z}) = \sum_{[\boldsymbol{l}_j]\in\mathcal{N}(\boldsymbol{D})} \boldsymbol{z}^{-\boldsymbol{l}_j^*} \boldsymbol{c}_j(\boldsymbol{z}^{\boldsymbol{D}}).$$

Without loss of generality, we may assume $\boldsymbol{l}_0 = \boldsymbol{0}$. Then the dot product is

$$\boldsymbol{r}(\boldsymbol{z}) \cdot \boldsymbol{v}(\boldsymbol{z}) = 1$$

for all $\boldsymbol{z}$ by (5.9). But

$$\boldsymbol{v}(\boldsymbol{z}) = \sum_{[\boldsymbol{l}_j]\in\mathcal{N}(\boldsymbol{D})} \boldsymbol{z}^{-\boldsymbol{l}_j^*} \boldsymbol{c}_j(\boldsymbol{z}^{\boldsymbol{D}}) \quad = \begin{pmatrix} H_0(\boldsymbol{z}) \\ H_1(\boldsymbol{z}) \\ \vdots \\ H_{N-1}(\boldsymbol{z}). \end{pmatrix}$$

Let $\boldsymbol{z}_0$ be a common zero of $\{H_i(\boldsymbol{z})\}$. Then

$$\boldsymbol{r}(\boldsymbol{z}_0) \cdot \boldsymbol{v}(\boldsymbol{z}_0) = 0 \neq 1$$

which leads to contradiction. □

Proposition 7. *Every common solution of the FIR analysis filters $H_i(\boldsymbol{z})$ has at least one zero component if and only if the system provide a PR for some sampling matrix and some FIR synthesis polyphase matrix.*

Proof. We may assume $H_i(\boldsymbol{z})$ are all polynomial by multiplying a least common monomial. To show the system provide a PR for some sampling matrix and some FIR synthesis polyphase matrix, again it is enough to show to show when sampling matrix $D = I$ (i.e. no downsampling and no downsampling within the filter bank), there exists some FIR synthesis filters to have a PR. Suppose if $H_i(\boldsymbol{z})$ have no common solution, then by Proposition 6 the system provide a PR for some sampling matrix and some polynomial synthesis polyphase matrix. Now consider $H_i(\boldsymbol{z})$ have common solutions. Let $f(\boldsymbol{z}) = z_1 z_2 ... z_M$. Then $f(s) = 0$ for any $s \in V(\{H_i(\boldsymbol{z})\})$. By Hilbert's Nullstellensatz Theorem [28], there exists a positive integer m such that $f(\boldsymbol{z})^m \in \langle H_0(\boldsymbol{z}), H_1(\boldsymbol{z}), ..., H_{N-1}(\boldsymbol{z})\rangle$. Therefore there exists polynomials $G_i(\boldsymbol{z})$ such that

$$\sum_{i=0}^{N-1} (z_1 z_2 ... z_M)^{-m} G_i(\boldsymbol{z}) H_i(\boldsymbol{z}) = 1.$$

This shows that the system provide PR for some sampling matrix and some FIR synthesis polyphase matrix. The other direction of the proof is similar the proof in Proposition 6. □

The last proposition addresses the first problem in the beginning of this chapter. To determine whether the system is perfectly reconstructable for given a set of analysis filters, we only need to examine the solution set of the analysis filters.

Example 18. *Let*

$$H_0(\boldsymbol{z}) = (1+z_1)(1+z_2), \quad H_1(\boldsymbol{z}) = (1-z_1)(1-z_1z_2),$$

$$H_2(\boldsymbol{z}) = (1-z_1)(z_1-z_2), \quad H_3(\boldsymbol{z}) = (1-z_2)(1-z_1z_2),$$

be the analysis filters. Determine whether is the system perfectly reconstructable or not?

Since $V(\{H_0(\boldsymbol{z}), H_1(\boldsymbol{z}), ..., H_3(\boldsymbol{z})\}) = \{(-1,-1)\}$, this implies it is not perfectly reconstructable. However, if we change $H_0(\boldsymbol{z}) = (1+pz_1)(1+qz_2)$ for some $p, q \neq 1$, then the system will become perfectly reconstructable.

Remark 4. *Suppose now consider $P > N$. Since the rank of the matrix $\boldsymbol{H}(\boldsymbol{z})$ is N, it is less than number of columns P. Therefore it is impossible to have a left inverse matrix if $P > N$. This provides us an upper bound of the search method.*

5.3.3 Search for Maximum Density Sampling Matrices

According to above Remark 4 and Hermite normal form, the search space for sampling matrices to determine the perfect reconstructability is a finite process. We propose an algorithm to answer the second problem mentioned in the beginning of the chapter.

Algorithm 8. The algorithm of finding a sampling matrix with maximal sampling factor and synthesis filters satisfying the PR condition.

Input: FIR analysis filters.

Output: sampling matrix $\boldsymbol{D}$ and FIR synthesis polyphase matrix together with the set $\mathcal{N}(\boldsymbol{D})$, if they exist.

1. Check the solution set of $H_i(\boldsymbol{z})$ satisfying the condition of Proposition 7. Otherwise there is no solution.

2. Set $P = N$.

3. Call Algorithm 7; Output FIR synthesis polyphase matrix, the set $\mathcal{N}(\boldsymbol{D})$ and a sampling matrix $\boldsymbol{D}$ if the inverse exists.

4. Set $P = P - 1$ and go to 3.

By Theorem 15, the maximum number of iterations is $\sum_{m=1}^{N} \sum_{q|m} q \cdot |C_{M-1}(q)|$ for $M \geq 2$. When $M = 2$, then $\sum_{m=1}^{N} \sum_{q|m} q = \frac{\pi^2}{12}N^2 + \mathrm{O}(N \ln N)$ by Handy and Weight [25]. To improve the performance, we can reduce the upper bound P by setting to $N - M$ due to the fact that the possibility of being a PR system is zero when $N - M < P$ [37].

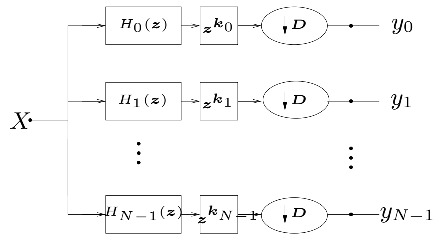


Figure 5.3: Multichannel convolution together with pure delays followed by downsampling D.

Example 19. *Let* $H_0(\boldsymbol{z}) = (1+z_1)(1+z_2), H_1(\boldsymbol{z}) = (1-z_1)(1-z_1z_2), H_2(\boldsymbol{z}) = (1-z_1)(z_1-z_2), H_3(\boldsymbol{z}) = (1-z_2)(1-z_1z_2), H_4(\boldsymbol{z}) = (1-z_2)(z_1-z_2), H_5(\boldsymbol{z}) = (1-z_1)(1-z_2)$ *be the analysis filters. By the Algorithm 8, we can show that the solution set of analysis filters is an empty set, which implies the system is polynomial perfectly reconstructable. We then obtain a sampling matrix* $\boldsymbol{D} = \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix}$ *with maximum sampling rate and synthesis matrix*

$$\boldsymbol{G}(\boldsymbol{z}) = \begin{pmatrix} -\frac{z_1}{8}+\frac{3}{8} & -\frac{z_1z_2}{24}-\frac{z_1}{12}+\frac{z_2}{8}+\frac{1}{4} & -\frac{z_1z_2}{12}-\frac{z_1}{24}+\frac{z_2}{4}+\frac{1}{8} \\ \frac{1}{4} & \frac{z_2}{12}+\frac{1}{2} & \frac{z_2}{6}+\frac{3}{4} \\ 0 & \frac{z_2}{3} & \frac{2z_2}{3} \\ \frac{1}{4} & \frac{z_1}{3}+\frac{z_2}{12}-\frac{1}{2} & \frac{2z_1}{3}+\frac{z_2}{6}-\frac{5}{4} \\ 0 & -\frac{2z_1}{3} & -\frac{4z_1}{3}+1 \\ \frac{z_1}{8}+\frac{1}{8} & \frac{z_1z_2}{24}-\frac{z_1}{4}-\frac{7z_2}{24}-\frac{1}{4} & \frac{z_1z_2}{12}-\frac{5z_1}{8}-\frac{7z_2}{12}+\frac{3}{8} \end{pmatrix}^T$$

satisfying the PR condition.

5.3.4 Systems with Pure Delays

For the simplicity, we assume that the sampling matrix is fixed in this subsection. We allow pure delays before the downsampling for each branch of analysis part shown in Fig. 5.3. We would like to develop algorithm to find a PR synthesis polyphase matrix with pure delays allowed. By using multirate identities [57], it is easy to see that we only need to test the invertibility of the delays $\{\boldsymbol{z}^{\boldsymbol{l}_0}, ..., \boldsymbol{z}^{\boldsymbol{l}_{P-1}}\}$ for each branch of analysis part where $\boldsymbol{z}^{\boldsymbol{l}_i} \in \mathcal{N}(\boldsymbol{D})$. We can fix the first channel with no delay because if the system contains the analysis part $\{\boldsymbol{z}^{\boldsymbol{k}_0} H_0(\boldsymbol{z}), \boldsymbol{z}^{\boldsymbol{k}_1} H_1(\boldsymbol{z}), ..., \boldsymbol{z}^{\boldsymbol{k}_{N-1}} H_{N-1}(\boldsymbol{z})\}$ provides a perfect reconstruction, then the system contains the analysis part $\{H_0(\boldsymbol{z}), \boldsymbol{z}^{\boldsymbol{k}_1-\boldsymbol{k}_0} H_1(\boldsymbol{z})..., \boldsymbol{z}^{\boldsymbol{k}_{N-1}-\boldsymbol{k}_0} H_{N-1}(\boldsymbol{z})\}$ also provides a perfect reconstruction. Now we can develop a PR synthesis polyphase matrix algorithm with pure delays allowed.

Algorithm 9 (PR Synthesis Matrix With Pure Delays)**.** *The computational algorithm for PR synthesis polyphase matrix with pure delays.*

Input: $\{H_0(\boldsymbol{z}), ..., H_{N-1}(\boldsymbol{z})\}$, $\mathcal{N}(\boldsymbol{D})$, *and* $\boldsymbol{D}$.

Output: $P \times N$ *inverse matrix* $\boldsymbol{G}(\boldsymbol{z})$ *and the delays* $\{\boldsymbol{z}^{\boldsymbol{k}_1}, ..., \boldsymbol{z}^{\boldsymbol{k}_{N-1}}\}$, *if they exist.*

1. *Let* $E = \{(\boldsymbol{k}_1, ..., \boldsymbol{k}_{N-1}) \mid k_i \in \mathcal{N}(\boldsymbol{D})\}$ *and set* $\tilde{H}_0(\boldsymbol{z}) = H_0(\boldsymbol{z})$.
2. *Let* $(\boldsymbol{k}_1, ..., \boldsymbol{k}_{N-1}) \in E$. *Set Set* $E := E - \{(\boldsymbol{k}_1, ..., \boldsymbol{k}_{N-1})\}$.
3. *Set* $\tilde{H}_i(\boldsymbol{z}) = \boldsymbol{z}^{\boldsymbol{k}_i} H_i(\boldsymbol{z})$ *for* $i = 1, ..., N-1$.
4. *Call Algorithm 6 to compute the analysis polyphase matrix* $\tilde{\boldsymbol{H}}(\boldsymbol{z})$ *of* $\tilde{H}_i(\boldsymbol{z})$.
5. *Call Algorithm 4; If* $\tilde{\boldsymbol{H}}(\boldsymbol{z})$ *is invertible, then output* $\tilde{\boldsymbol{G}}(\boldsymbol{z})$ *and* $\{\boldsymbol{z}^{\boldsymbol{k}_1}, ..., \boldsymbol{z}^{\boldsymbol{k}_{N-1}}\}$.
6. *If* E *is not empty, then goto 2. Otherwise, there is no solution.*

By the design of the algorithm, the total number of iterations is P^{N-1}.

5.4 Frame Analysis

5.4.1 Inner Products

The collected data in multichannel acquisition can be treated as inner products in appropriate vector spaces. Let $\boldsymbol{n} = (n_1, ..., n_M) \in \mathbb{Z}^M$ where M be a fixed positive integer. Let $x = (x[\boldsymbol{n}])_{\boldsymbol{n}\in\mathbb{Z}^M}$ be a M-multidimensional infinite sequence where $x[\boldsymbol{n}]$ is in $\mathbb{C}$. Let $\mathcal{W}$ be a vector space containing all M-multidimensional infinite sequences. We define an inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{W}$ over $\mathbb{C}$ such that

$$\langle x, x' \rangle = \sum_{\boldsymbol{n}\in\mathbb{Z}^M} x[\boldsymbol{n}] x'^*[\boldsymbol{n}]. \tag{5.10}$$

Then $\ell^2(\mathbb{Z}^M)$ is defined as

$$\ell^2(\mathbb{Z}^M) = \{x \in \mathcal{W} \mid \|x\|_E^2 < \infty\}.$$

Remark 5. *In the later context, we will use* $\|\cdot\|_2$ *to denote the matrix two norm. To avoid the confusion of the notation, we denote* $\|x\|_E := \sqrt{\sum_{\boldsymbol{n}\in\mathbb{Z}^M} x[\boldsymbol{n}] x^*[\boldsymbol{n}]}$ *to be an Euclidean Norm.*

Instead of restricting to a infinite sequence, we define an inner product $\langle \cdot, \cdot \rangle$ on a column of infinite sequences $\mathcal{W}^N$ over $\mathbb{C}$ such that

$$\langle \boldsymbol{x}, \boldsymbol{x}' \rangle = \sum_{i=0}^{N-1} \sum_{\boldsymbol{n}\in\mathbb{Z}^M} x_i[\boldsymbol{n}] x_i'^*[\boldsymbol{n}]. \tag{5.11}$$

On the other hand, we can also define an inner product on a Laurent polynomial matrix. Let $\mathcal{R} = \mathbb{C}[z_1, ..., z_M]$ where M be a fixed positive integer. We define an inner product $\langle \cdot, \cdot \rangle$ on a column vector $\mathcal{R}^P$ over $\mathbb{C}$ such that

$$\langle \boldsymbol{X}(\boldsymbol{z}), \boldsymbol{X}'(\boldsymbol{z}) \rangle = \int_{[0,1)^M} \boldsymbol{X}'(e^{2\pi j\boldsymbol{\theta}})^H \boldsymbol{X}(e^{2\pi j\boldsymbol{\theta}})\, d\boldsymbol{\theta} \tag{5.12}$$

where H is a conjugate transpose. Similarly we define an inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{R}^{P\times N}$ over $\mathbb{C}$ such that

$$\langle \boldsymbol{X}(\boldsymbol{z}), \boldsymbol{X}'(\boldsymbol{z}) \rangle = \int_{[0,1)^M} \mathrm{tr}\Big(\boldsymbol{X}'(e^{2\pi j \boldsymbol{\theta}})^H \boldsymbol{X}(e^{2\pi j \boldsymbol{\theta}}) \Big)\, d\boldsymbol{\theta} \tag{5.13}$$

where H is a conjugate transpose. To overload a notation, we denote $\|\cdot\|_E := \sqrt{\langle \cdot, \cdot \rangle}$ for all inner products defined above.

5.4.2 Filter Bank and Frames

The z-transform of impulse response of analysis part $h_k[\boldsymbol{n}]$ and synthesis part $g_k[\boldsymbol{n}]$ yield the transfer functions $H_k(\boldsymbol{z})$ and $G_k(\boldsymbol{z})$ respectively. The subband signals of analysis filter bank are given by

$$\begin{aligned} y_k[\boldsymbol{m}] &= (\mathrm{H}x)[k, \boldsymbol{m}] \\ &= \sum_{\boldsymbol{n}\in\mathbb{Z}^M} x[\boldsymbol{n}] h_k[\boldsymbol{D}\boldsymbol{m} - \boldsymbol{n}] \\ &= \langle x, h^{-*}_{k,\boldsymbol{m}} \rangle \end{aligned} \tag{5.14}$$

for $k = 0, 1, ..., N-1$ where $h^{-*}_{k,\boldsymbol{m}}[\boldsymbol{n}] = h^*_k[-(\boldsymbol{n} - \boldsymbol{D}\boldsymbol{m})]$ for $k = 0, 1, ..., N-1$ and $\boldsymbol{m} \in \mathbb{Z}^M$, and H is the FB analysis operator. The reconstructed signal is

$$\begin{aligned} \hat{x}[\boldsymbol{n}] &= (\mathrm{G}\boldsymbol{y})[\boldsymbol{n}] \\ &= \sum_{(k,\boldsymbol{m})\in K} y_k[\boldsymbol{m}] g_k[\boldsymbol{n} - \boldsymbol{D}\boldsymbol{m}] \\ &= \sum_{(k,\boldsymbol{m})\in K} y_k[\boldsymbol{m}] g_{k,\boldsymbol{m}}[\boldsymbol{n}] \end{aligned} \tag{5.15}$$

where $g_{k,\boldsymbol{m}}[\boldsymbol{n}] = g_k[\boldsymbol{n} - \boldsymbol{D}\boldsymbol{m}]$, $\boldsymbol{y}[\boldsymbol{m}] = \begin{pmatrix} y_0[\boldsymbol{m}] & y_1[\boldsymbol{m}] & \cdots & y_{N-1}[\boldsymbol{m}] \end{pmatrix}^T$, and G is the FB synthesis operator [16].

Definition 28. *[5] Define $\mathcal{Z}^{(N)}_{\mathcal{L}} : \mathcal{W}^N \to \mathcal{R}^{P\times N}$ to be a Zak transform operator such that*

$$(a_0, ..., a_{N-1}) \mapsto \begin{pmatrix} A_{0,0}(\boldsymbol{z}) & \cdots & A_{0,N-1}(\boldsymbol{z}) \\ \vdots & \cdots & \vdots \\ A_{P-1,0}(\boldsymbol{z}) & \cdots & A_{P-1,N-1}(\boldsymbol{z}) \end{pmatrix}$$

where $A_{i,j}(\boldsymbol{z}) = \sum_{\boldsymbol{m}\in\mathbb{Z}^M} a_i[\boldsymbol{D}\boldsymbol{m} + \boldsymbol{k}_j]\boldsymbol{z}^{-\boldsymbol{m}}$ where $\{[\boldsymbol{k}_0], ..., [\boldsymbol{k}_{P-1}]\} = \mathcal{L}$.

Remark 6. *By Parseval's theorem, we have*

$$\|\boldsymbol{X}(\boldsymbol{z})\|_E^2 = \langle \mathcal{Z}^{(1)}_{\mathcal{N}(\boldsymbol{D})} x, \mathcal{Z}^{(1)}_{\mathcal{N}(\boldsymbol{D})} x \rangle = \|x\|_E^2.$$

Also,

$$\|\boldsymbol{G}(\boldsymbol{z})\|_E^2 = \langle \mathcal{Z}^{(N)}_{\mathcal{N}(\boldsymbol{D})} \boldsymbol{g}, \mathcal{Z}^{(N)}_{\mathcal{N}(\boldsymbol{D})} \boldsymbol{g} \rangle = \|\boldsymbol{g}\|_E^2 \text{ and}$$
$$\|\boldsymbol{Y}(\boldsymbol{z})\|_E^2 = \langle \mathcal{Z}^{(N)}_{\{\boldsymbol{0}\}} \boldsymbol{y}, \mathcal{Z}^{(N)}_{\{\boldsymbol{0}\}} \boldsymbol{y} \rangle = \|\boldsymbol{y}\|_E^2$$

where $\boldsymbol{g} = (g_0, ..., g_{N-1})^T$.

Let $\mathcal{H}$ be a separable Hilbert space, with the inner product $\langle \cdot, \cdot \rangle$ chosen to be linear in the first entry.

Definition 29. *[13] A sequence* $\{\varphi_{i,\boldsymbol{j}}\}_{(i,\boldsymbol{j})\in K}$ *in* $\mathcal{H}$ *is a frame for* $\ell^2(\mathbb{Z}^M)$ *if there exist constants* $A_\varphi, B_\varphi > 0$ *such that*

$$A_\varphi \|x\|_E^2 \leq \sum_{(i,\boldsymbol{j})\in K} |\langle x, \varphi_{i,\boldsymbol{j}} \rangle|^2 \leq B_\varphi \|x\|_E^2 \tag{5.16}$$

for every $x \in \ell^2(\mathbb{Z}^M)$. *The numbers* A_φ *and* B_φ *are called frame bounds for* $\{\varphi_{i,\boldsymbol{j}}\}$. *Without loss of generality, we may assume that* A_φ *and* B_φ *are the supremum over all lower frame bounds and the infimum over all upper frame bounds respectively.*

Definition 30. *A uniform filter bank associated with* $\{h^{-*}_{k,\boldsymbol{m}}\}_{(k,\boldsymbol{m})\in K}$ *is called the uniform filter bank frame (UFBF) if* $\{h^{-*}_{k,\boldsymbol{m}}\}_{(k,\boldsymbol{m})\in K}$ *is a frame for any* $x \in \ell^2(\mathbb{Z}^M)$.

If the filter bank satisfies perfect reconstruction, i.e., $\hat{x}[\boldsymbol{n}] = x[\boldsymbol{n}]$, then, by (5.14) and (5.15), we have

$$x[\boldsymbol{n}] = \sum_{(k,\boldsymbol{m})\in K} \langle x, h^{-*}_{k,\boldsymbol{m}} \rangle g_{k,\boldsymbol{m}}[\boldsymbol{n}] \tag{5.17}$$

5.4.3 Perturbation of Subband Signals and Filters

Definition 31. *We define* $\|\cdot\|_S$ *to be the essential supremum of two norm on the unit sphere. i.e.*

$$\|\boldsymbol{R}(\boldsymbol{z})\|_S := \operatorname{ess\,sup}_{\boldsymbol{\theta}\in[0,1)^M} \|\boldsymbol{R}(e^{2\pi j\boldsymbol{\theta}})\|_2.$$

Lemma 6. *Suppose* $\{h^{-*}_{k,\boldsymbol{m}}\}_{(k,\boldsymbol{m})\in K}$ *is a UFBF with the optimal upper frame bounds* B_h *and* $\{g_{k,\boldsymbol{m}}\}_{(k,\boldsymbol{m})\in K}$ *is a dual with the optimal upper frame bounds* B_g. *Then* $B_h = \|\boldsymbol{H}(\boldsymbol{z})\|_S^2$ *and* $B_g = \|\boldsymbol{G}(\boldsymbol{z})^H\|_S^2$.

Proof. By Lemma 28 and Theorem 1.2.1 of [13]. □

Remark 7. *The nonzero eigenvalues* $\lambda'_n(\boldsymbol{\theta})$ *of* $\boldsymbol{G}(e^{2\pi j\boldsymbol{\theta}})\boldsymbol{G}(e^{2\pi j\boldsymbol{\theta}})^H$ *and* $\boldsymbol{G}(e^{2\pi j\boldsymbol{\theta}})^H\boldsymbol{G}(e^{2\pi j\boldsymbol{\theta}})$ *are the same.*

Lemma 7. *Suppose a dual of a UFBF is* $\{g_{k,\boldsymbol{m}}\}_{(k,\boldsymbol{m})\in K}$ *with the optimal upper frame bounds* B_g. *Then*

$$\|\hat{\boldsymbol{x}}\|_E^2 = \langle \mathrm{G}\boldsymbol{y}, \mathrm{G}\boldsymbol{y}\rangle = \langle \mathrm{G}^*\mathrm{G}\boldsymbol{y}, \boldsymbol{y}\rangle \leq B_g \|\boldsymbol{y}\|_E^2. \tag{5.18}$$

Proof. Followed by Remark 7. ☐

Corollary 2. *Let* $\boldsymbol{G}(\boldsymbol{z})$ *be a PR synthesis polyphase matrix for a given analysis polyphase matrix* $\boldsymbol{H}(\boldsymbol{z})$. *Let* $\Delta\boldsymbol{y}$ *be a perturbation of subband signals. Then*

$$\|\hat{x} - x\|_E \leq \|\boldsymbol{G}(\boldsymbol{z})\|_S \|\Delta\boldsymbol{y}\|_E.$$

Corollary 3. *Let* $\boldsymbol{H}(\boldsymbol{z})$ *and* $\boldsymbol{G}(\boldsymbol{z})$ *be the corresponding polyphase matrices of* h *and* g *respectively. Let* $\Delta\boldsymbol{H}(\boldsymbol{z})$ *be a perturbation of analysis polyphase matrix. Then*

$$\|\hat{x} - x\|_E \leq \|\boldsymbol{G}(\boldsymbol{z})\|_S \|\Delta\boldsymbol{H}(\boldsymbol{z})\|_S \|x\|_E.$$

To reduce the noise level from the output with respect to $\|\cdot\|_E$, it would be ideal to find an optimal $\boldsymbol{G}(\boldsymbol{z})$ among all inverses with respect to $\|\cdot\|_S$. However, in the later subsections. we will show that this is not practical to look for such optimal solution.

5.4.4 Finite Support

In this subsection, we will show that even though the para-pseudoinverse $\boldsymbol{H}^{\dagger}(\boldsymbol{z})$ is the optimal solution among the inverses with respect to the Euclidean norm $\|\cdot\|_E$ and essential supremum two norm on unit sphere $\|\cdot\|_S$, it is not a good choice in practice because the support of $\boldsymbol{H}^{\dagger}(\boldsymbol{z})$ is infinite in most cases. We provide an alternative method to design a system such that we almost surely guarantee that we have an inverse with finite support.

Definition 32. *The paraconjugate of* $\boldsymbol{H}(\boldsymbol{z})$ *is defined as*

$$\tilde{\boldsymbol{H}}(\boldsymbol{z}) := \boldsymbol{H}^H(\boldsymbol{z}^{-1})$$

where $\boldsymbol{H}^H(\boldsymbol{z})$ *denotes transpose of* $\boldsymbol{H}(\boldsymbol{z})$ *followed by the conjugation of the coefficients within* $\boldsymbol{H}^T(\boldsymbol{z})$. *Let* $\boldsymbol{S}(\boldsymbol{z}) = \tilde{\boldsymbol{H}}(\boldsymbol{z})\boldsymbol{H}(\boldsymbol{z})$. *The para-pseudoinverse of* $\boldsymbol{H}(\boldsymbol{z})$ *is defined as*

$$\boldsymbol{H}^{\dagger}(\boldsymbol{z}) = \boldsymbol{S}(\boldsymbol{z})^{-1}\tilde{\boldsymbol{H}}(\boldsymbol{z})$$

Lemma 8. *Suppose the para-pseudoinverse $\boldsymbol{H}^{\dagger}(\boldsymbol{z})$ is well defined. Then $\boldsymbol{H}^{\dagger}(\boldsymbol{z})$ is a synthesis polyphase matrix and $\boldsymbol{H}^{\dagger}(\boldsymbol{z})$ is the minimum with respect to $\|\cdot\|_S$ among all the synthesis polyphase matrices. Suppose $h_k^{\dagger}[n]$ is impulse responses corresponding to $\boldsymbol{H}^{\dagger}(\boldsymbol{z})$. Then $h_k^{\dagger}[n] = S^{-1}h_k^{-*}[n]$ where $h_k^{-*}[n] = h_k^*[-n]$. Therefore, the optimal upper frame bound $B_{S^{-1}h}$ is the minimum among all the dual frame bounds.*

Proof. For every $\boldsymbol{\theta}$ is on unit sphere, an optimal solution is $\boldsymbol{H}^{\dagger}(e^{2\pi j\boldsymbol{\theta}})$ among all inverses with respect to $\|\cdot\|_2$. Therefore, $\boldsymbol{H}^{\dagger}(\boldsymbol{z})$ is the minimum with respect to $\|\cdot\|_S$ among all the synthesis polyphase matrices. The second part is followed by the fact that $\mathcal{Z}^{(N)}_{\mathcal{N}(\boldsymbol{D})}S^{-1} = \boldsymbol{S}(\boldsymbol{z})^{-1}\mathcal{Z}^{(N)}_{\mathcal{N}(\boldsymbol{D})}$. (See [6]) □

Lemma 9. *Suppose the para-pseudoinverse $\boldsymbol{H}^{\dagger}(\boldsymbol{z})$ is well defined. Then $\boldsymbol{H}^{\dagger}(\boldsymbol{z})$ is minimum with respect to $\|\cdot\|_E$ among all the synthesis polyphase matrices.*

Proof. Since every synthesis polyphase matrix can be expressed as

$$\boldsymbol{G}(\boldsymbol{z}) = \boldsymbol{H}^{\dagger}(\boldsymbol{z}) + \boldsymbol{A}(\boldsymbol{z})(\boldsymbol{I} - \boldsymbol{H}(\boldsymbol{z})\boldsymbol{H}^{\dagger}(\boldsymbol{z})) \tag{5.19}$$

where $\boldsymbol{A}(\boldsymbol{z})$ is an arbitrary $P \times N$ matrix. Then $\|\boldsymbol{G}(\boldsymbol{z})\|_E = \|\boldsymbol{H}^{\dagger}(\boldsymbol{z})\|_E + \|\boldsymbol{A}(\boldsymbol{z})(\boldsymbol{I} - \boldsymbol{H}(\boldsymbol{z})\boldsymbol{H}^{\dagger}(\boldsymbol{z}))\|_E$ because $\langle \boldsymbol{H}^{\dagger}(\boldsymbol{z}), \boldsymbol{A}(\boldsymbol{z})(\boldsymbol{I} - \boldsymbol{H}(\boldsymbol{z})\boldsymbol{H}^{\dagger}(\boldsymbol{z}))\rangle = 0$. □

Lemma 10. *[40] Let $f(\boldsymbol{z})$ be a nonzero polynomial in $\mathbb{C}[z_1, ..., z_M]$. Then $1/f(\boldsymbol{z})$ is formal power series in $\mathbb{C}[[z_1, ..., z_M]]$. Furthermore, if $f(\boldsymbol{z})$ is not monomial, then $1/f(\boldsymbol{z})$ is not a Laurent polynomial.*

Corollary 4. *Let $\boldsymbol{H}(\boldsymbol{z})$ be an analysis polyphase matrix with a full rank. If $\det \boldsymbol{S}(\boldsymbol{z})$ is not a monomial, then $\boldsymbol{H}^{\dagger}(\boldsymbol{z})$ is not Laurent polynomial matrix. i.e. $S^{-1}h_k^{-*}$ do not have a finite support.*

Although $\boldsymbol{H}^{\dagger}(\boldsymbol{z})$ is is the minimum with respect to $\|\cdot\|_S$ or $\|\cdot\|_E$ among all PR synthesis polyphase matrices, $\boldsymbol{H}^{\dagger}(\boldsymbol{z})$ may not be the practical PR synthesis polyphase matrix to use. The reason is that the possibility of $\det \boldsymbol{S}(\boldsymbol{z})$ being a monomial is zero. So $\boldsymbol{H}^{\dagger}(\boldsymbol{z})$ usually is not FIR. Therefore, when we design the system, we should make sure that we could have the finite support synthesis impulse responses with a small frame bounds. Theorem 9 and Theorem 10 tell us that if $N - P \geq M$, the PR synthesis impulse responses almost surely have a finite support. On the other hands, if $N - P < M$, the PR synthesis impulse responses almost surely do not have a finite support. So we should design our system such that $N - P \geq M$.

5.4.5 Essential Supremum Two Norm Estimation

Lemma 11. *[6] Let $\boldsymbol{H}(\boldsymbol{z})$ and $\boldsymbol{G}(\boldsymbol{z})$ be an analysis polyphase matrix and a synthesis polyphase matrix respectively. Then*

$$\frac{1}{\sqrt{P}}\|\boldsymbol{h}\|_E \leq \|\boldsymbol{H}(\boldsymbol{z})\|_S$$

where $\boldsymbol{h} = (h_0, ..., h_{N-1})^T$*. Similarly*

$$\frac{1}{\sqrt{P}}\|\boldsymbol{g}\|_E \leq \|\boldsymbol{G}(\boldsymbol{z})\|_S$$

where $\boldsymbol{g} = (g_0, ..., g_{N-1})^T$*.*

Remark 8. *Since it may not be practical to find* $\|\boldsymbol{G}(\boldsymbol{z})\|_S$*, one would like to estimate* $\|\boldsymbol{G}(\boldsymbol{z})\|_S$*. However, by Lemma 11, we know that* $\frac{1}{\sqrt{P}}\|\boldsymbol{g}\|_E \leq \|\boldsymbol{G}(\boldsymbol{z})\|_S$*. Since* $\|\boldsymbol{g}\|_E$ *is the square root of the sum of the square of the coefficients, it is easy to evaluate the value. Therefore if* $\frac{1}{\sqrt{P}}\|\boldsymbol{g}\|_E$ *is large, then we know that* $\|\boldsymbol{G}(\boldsymbol{z})\|_S$ *is also large, which implies that the system is unstable.*

By Lemma 11, we can find a lower bound of $\|\boldsymbol{G}(\boldsymbol{z})\|_S$. We also want to have a rough upper bound too.

Lemma 12. *Let* $\boldsymbol{H}(\boldsymbol{z})$ *and* $\boldsymbol{G}(\boldsymbol{z})$ *be an analysis polyphase matrix and a synthesis polyphase matrix respectively. Suppose* $\sum_{\boldsymbol{k}\in Q_{i,j}} h_{i,j,\boldsymbol{k}}\boldsymbol{z}^{\boldsymbol{k}}$ *be the* (i,j) *entry of* $\boldsymbol{H}(\boldsymbol{z})$ *for some* $Q_{i,j}$*. Let* $\boldsymbol{H}' = (\sum_{\boldsymbol{k}\in Q_{i,j}} |h_{i,j,\boldsymbol{k}}|)$ *be an* $N \times P$ *matrix over* $\mathbb{R}$*. Similarly, we denote* $\boldsymbol{G}'$ *in the same fashion. Then*

$$\|\boldsymbol{H}(\boldsymbol{z})\|_S \leq \sqrt{P}\|\boldsymbol{H}'\|_1 \text{ and}$$
$$\|\boldsymbol{G}(\boldsymbol{z})\|_S \leq \sqrt{N}\|\boldsymbol{G}'\|_1 = \sqrt{N} \max_{i=0,\ldots,N-1} \|g_i\|_1.$$

Moreover,

$$\|\boldsymbol{G}(\boldsymbol{z})\|_S \leq \sqrt{P}\|\boldsymbol{G}'\|_\infty \text{ and}$$
$$\|\boldsymbol{H}(\boldsymbol{z})\|_S \leq \sqrt{N}\|\boldsymbol{H}'\|_\infty = \sqrt{N} \max_{i=0,\ldots,N-1} \|h_i\|_1.$$

Proof. Followed by [30, p.314] and by the definition of one norm $\|\cdot\|_1$ and $\|\cdot\|_\infty$. □

5.4.6 Optimization of Synthesis FIR Polyphase Matrices

Although it is ideal to find an inverse by minimizing FIR synthesis filters for a given support of $\boldsymbol{A}(\boldsymbol{z})$ shown in (3.6) with respect to $\|\cdot\|_S$, it is impractical to find such optimal solution. However, minimizing $\|\boldsymbol{g}\|_E$ among FIR synthesis filters for the given support of $\boldsymbol{A}(\boldsymbol{z})$ is feasible in practice and it provide a necessary condition to have stable system by Remark 8. To minimize $\|\boldsymbol{g}\|_E$ given above, it is same to minimize $\|\boldsymbol{G}(\boldsymbol{z})\|_E$ among synthesis FIR polyphase matrices for the given support of $\boldsymbol{A}(\boldsymbol{z})$ by Remark 6. Here, we provide an algorithm to find an optimal inverse with respect to $\|\cdot\|_E$.

Algorithm 10 (Euclidean Norm Optimal Inverse)**.** The computational algorithm for a left inverse matrix by minimizing with respect $\|\cdot\|_E$ for a given support $\{Q_{i,j}\}$ of $\boldsymbol{A}(\boldsymbol{z})$ where $A_{i,j}(\boldsymbol{z}) = \sum_{\boldsymbol{k}\in Q_{i,j}} a_{i,j,\boldsymbol{k}}\boldsymbol{z}^{\boldsymbol{k}}$.
Input: $N \times P$ *matrix* $\boldsymbol{H}(\boldsymbol{z})$ *with* M *variables.*

Output: an optimal inverse $P \times N$ matrix $\boldsymbol{G}(\boldsymbol{z})$ with respect to the given support.

1. *Compute a particular inverse $\tilde{\boldsymbol{G}}(\boldsymbol{z})$ by Algorithm 4.*
2. *Compute $\boldsymbol{G}(\boldsymbol{z})$ with respect to $a_{i,j,\boldsymbol{k}}$.*
3. *Compute $\|\boldsymbol{G}(\boldsymbol{z})\|_E$.*
4. *Set all partial derivatives of $\|\boldsymbol{G}(\boldsymbol{z})\|_E$ with respect to $a_{i,j,\boldsymbol{k}}$ to zero and solve the linear equations.*
5. *Back substitute an optimal solution $\tilde{\boldsymbol{A}}(\boldsymbol{z})$ in $\boldsymbol{G}(\boldsymbol{z})$.*
6. *Output $\boldsymbol{G}(\boldsymbol{z})$.*

Proposition 8. *Suppose that the measured data $\boldsymbol{H}(\boldsymbol{z})\boldsymbol{X}(\boldsymbol{z})$ is contaminated by additive white Gaussian noises $\boldsymbol{\epsilon}$ with zero mean and power density σ^2. Suppose $\mathcal{G}$ is a given set of inverses of $\boldsymbol{H}(\boldsymbol{z})$. Then the minimum mean square error for the set $\mathcal{G}$ is*

$$\mathrm{MMSE}(\mathcal{G}) = \frac{\sigma^2}{P} \min_{\boldsymbol{G}(\boldsymbol{z})\in\mathcal{G}} \|\boldsymbol{G}(\boldsymbol{z})\|_E^2.$$

Proof. The reconstruction error is $\boldsymbol{\eta} = \boldsymbol{G}(\boldsymbol{z})\boldsymbol{\epsilon}$. Then the covariance of $\boldsymbol{\eta}$ is $R_{\boldsymbol{\eta}\boldsymbol{\eta}} = \mathbb{E}[\boldsymbol{\eta}\boldsymbol{\eta}^H] = \sigma^2\mathbb{E}[\boldsymbol{G}(\boldsymbol{z})\boldsymbol{G}(\boldsymbol{z})^H]$. The minimum mean square error for the set $\mathcal{G}$ is

$$\begin{aligned}\mathrm{MMSE}(\mathcal{G}) &= \min_{\boldsymbol{G}(\boldsymbol{z})\in\mathcal{G}} \left\{\frac{1}{P}\sum_{i=1}^{P}\mathbb{E}[\eta_i^2]\right\} \\ &= \frac{1}{P}\min_{\boldsymbol{G}(\boldsymbol{z})\in\mathcal{G}}\left\{\mathrm{tr}(R_{\boldsymbol{\eta}\boldsymbol{\eta}})\right\} \\ &= \frac{\sigma^2}{P}\min_{\boldsymbol{G}(\boldsymbol{z})\in\mathcal{G}}\left\{\int_{[0,1)^M}\mathrm{tr}\left(\boldsymbol{G}(e^{2\pi j\boldsymbol{\theta}})\boldsymbol{G}(e^{2\pi j\boldsymbol{\theta}})^H\right)d\boldsymbol{\theta}\right\} \\ &= \frac{\sigma^2}{P}\min_{\boldsymbol{G}(\boldsymbol{z})\in\mathcal{G}}\|\boldsymbol{G}(\boldsymbol{z})\|_E^2. \qquad (5.20)\end{aligned}$$

□

This implies that Algorithm 10 gives us the PR FIR synthesis polyphase matrix with the minimum mean square error for a given support of $\boldsymbol{A}(\boldsymbol{z})$ when the system is corrupted by Gaussian noises.

By Lemma 12 and Corollary 2, we know $\|\hat{x} - x\|_E \le \sqrt{N}\max_{i=0,\dots,N-1}\|g_i\|_1\|\Delta\boldsymbol{y}\|_E$. Instead of minimizing with respect to $\|\cdot\|_E$, we can minimize $\max_{i=0,\dots,N-1}\|g_i\|_1$ (i.e. $\max_{i=0,\dots,N-1}\sum_{\boldsymbol{m}\in\mathbb{Z}^M}|g_{i,\boldsymbol{m}}|$). It is equivalent to

minimize w subject to

$$w - \sum_{\boldsymbol{m}\in\mathbb{Z}^M}|g_{i,\boldsymbol{m}}| \ge 0 \text{ for } i = 0, \dots, N-1.$$

Though this problem does not look like an LP problem, it may be converted into the following form:

$$\text{minimize } w \text{ subject to}$$

$$w - \sum_{\boldsymbol{m} \in \mathbb{Z}^M} e_{i,\boldsymbol{m}} \geq 0 \text{ for } i = 0, ..., N-1,$$

$$e_{i,\boldsymbol{m}} + g_{i,\boldsymbol{m}} \geq 0 \text{ for } i = 0, ..., N-1,\ \boldsymbol{m} \in \mathbb{Z}^M,$$

$$e_{i,\boldsymbol{m}} - g_{i,\boldsymbol{m}} \geq 0 \text{ for } i = 0, ..., N-1,\ \boldsymbol{m} \in \mathbb{Z}^M.$$

Therefore this is a linear optimization problem and can be solved by using linear programming.

Algorithm 11 (1-Norm Optimal Inverse)**.** The computational algorithm for a left inverse matrix by minimizing $\max_{i=0,...,N-1} \|g_i\|_1$ for a given support $\{Q_{i,j}\}$ of $\boldsymbol{A}(\boldsymbol{z})$ where $A_{i,j}(\boldsymbol{z}) = \sum_{\boldsymbol{k} \in Q_{i,j}} a_{i,j,\boldsymbol{k}} \boldsymbol{z}^{\boldsymbol{k}}$.

Input: $N \times P$ matrix $\boldsymbol{H}(\boldsymbol{z})$ with M variables.

Output: an optimal inverse $P \times N$ matrix $\boldsymbol{G}(\boldsymbol{z})$ with respect to the given support.

1. Compute a particular inverse $\tilde{\boldsymbol{G}}(\boldsymbol{z})$ by Algorithm 4.

2. Compute $\boldsymbol{G}(\boldsymbol{z})$ with respect to $a_{i,j,\boldsymbol{k}}$.

3. Use linear programming to find out an optimal solution $\tilde{\boldsymbol{A}}(\boldsymbol{z})$.

5. Back substitute the optimal solution $\tilde{\boldsymbol{A}}(\boldsymbol{z})$ in $\boldsymbol{G}(\boldsymbol{z})$.

6. Output $\boldsymbol{G}(\boldsymbol{z})$.

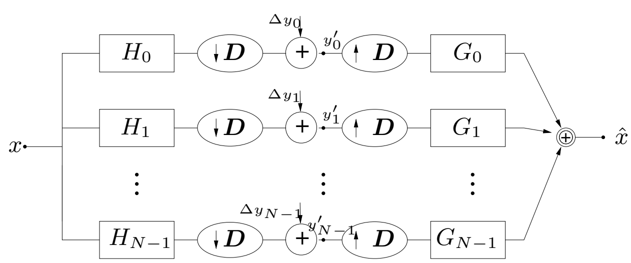
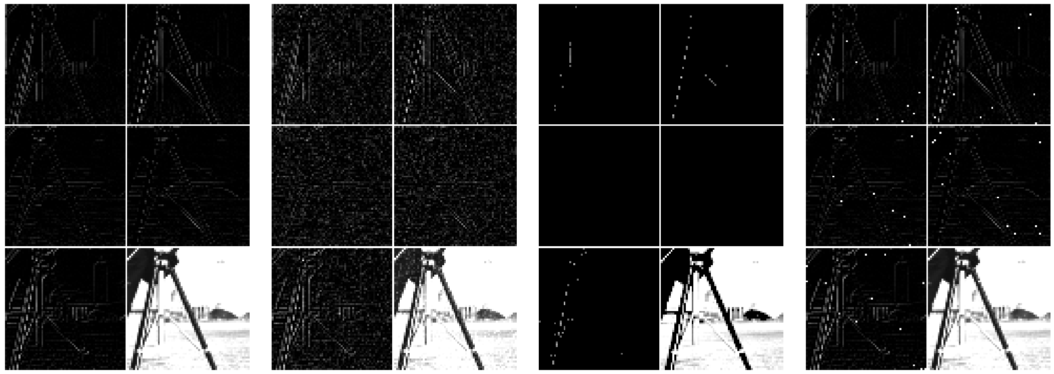


Figure 5.4: An N-channel filter bank with possible additive noise

Example 20. *We demonstrate a simulation of* 6*-channel filter bank with different types of additive noises.*



(a) (b) (c) (d)

Figure 5.5: The origin subband signals and the different additive noise subband signals. (a) The clear subband signals (b) Additive Gaussian noise with zero mean and variance $\sigma = 0.01$ (c) Eliminating 85000 of insignificant coefficients (d) Additive salt and pepper noise with noise densities $d = 0.005$.

Our input signals is the cameraman image of size 256×256*. Given analysis filters*

$$H_0(\boldsymbol{z}) = 0.5(1 - z_1)(1 - z_1 z_2),$$

$$H_1(\boldsymbol{z}) = 0.5(1 - z_1)(z_1 - z_2),$$

$$H_2(\boldsymbol{z}) = 0.5(1 - z_2)(1 - z_1 z_2),$$

$$H_3(\boldsymbol{z}) = 0.5(1 - z_2)(z_1 - z_2),$$

$$H_4(\boldsymbol{z}) = 0.5(1 - z_1^2 z_2)(1 - z_2^2 z_1)$$

$$H_5(\boldsymbol{z}) = 0.5(1 + z_1)(1 + z_2)$$

and a sampling matrix $\boldsymbol{D} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ *and the support of* $\boldsymbol{A}(\boldsymbol{z})$ *given in (3.6) is* $\{Q_{i,j}\}$ *where* $Q_{i,j} = \{\boldsymbol{0}\}$ *for all* i, j*. Let* Δy_i *be additive noise on each row independently. Then we can represent* y_i' *shown in Fig. 5.4, in the polyphase domain, as*

$$\boldsymbol{Y}'(\boldsymbol{z}) = \boldsymbol{H}(\boldsymbol{z})\boldsymbol{X}(\boldsymbol{z}) + \Delta\boldsymbol{Y}(\boldsymbol{z})$$

where $\Delta\boldsymbol{Y}(\boldsymbol{z})$ *be a polyphase matrix of* Δy_i*. We want to compare with reconstruction performance in* MSE *between Algorithm 3, 11, and 10. We show the subband signals in Fig. 5.5(b) - 5.5(d) with different types of noises. The reconstruction images are shown in Fig. 5.6(b) - 5.6(i). We find out that for all different noises, Algorithm 3 has the worst performance in* MSE *among the other algorithms. For Algorithm 10, it has the best performance in* MSE *with Gaussian noises and salt and pepper noises shown in Fig 5.7(a)*

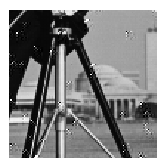
(a)

(b) (c) (d)

(e) (f) (g)

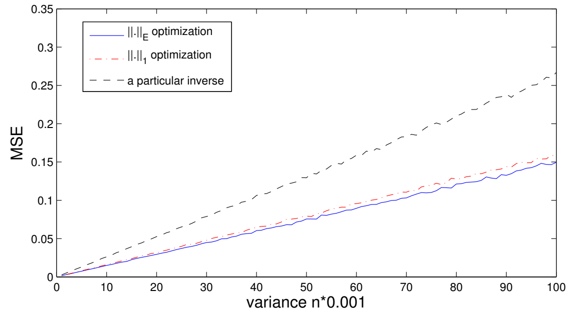
(h) (i) (j)

Figure 5.6: The origin image and the reconstruction outputs with additive noises. (a) The original image. (b)-(d) Algorithm 3, 10, and 11 with Gaussian noise ($\sigma = 0.01$), MSE=0.0259, 0.0147, and 0.0157. (e)-(g) Algorithm 3, 10, and 11 with eliminating 85000 of insignificant coefficients, MSE=0.0082, 0.0063, and 0.0060. (h)-(j) Algorithm 3, 10, and 11 with salt and pepper noise (noise density 0.005), MSE=0.0145, 0.0058, and 0.0062.

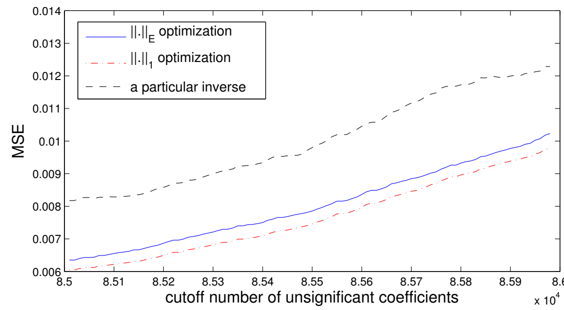
and Fig 5.7(c). For Algorithm 11, it has the best performance in MSE *if we eliminate different numbers of insignificant coefficients shown in Fig 5.7(b).*

5.5 Conclusion

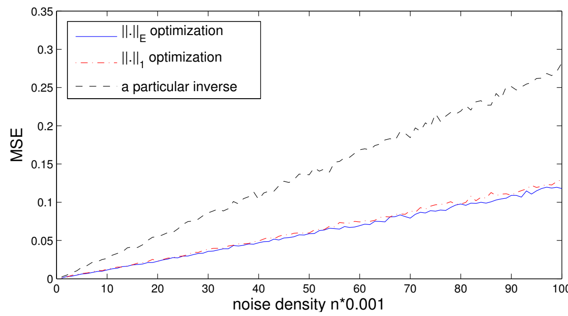
In this chapter, we study the theory and algorithms for the optimal use of multidimensional signal reconstruction from multichannel acquisition using a filter bank setup. From Propsition 6 and Proposition 7, we address the necessary and sufficient condition on the analysis filters to have a PR system with some



(a)



(b)



(c)

Figure 5.7: MSE of the reconstruction errors. (a) Additive Gaussian noise with zero mean and different levels of variance (b) Eliminate the different numbers of insignificant coefficients (c) Additive salt and pepper noise with different noise densities.

sampling matrix and some synthesis matrix. Using Hermite and Smith normal form, we propose Algorithms 8 together with Algorithm 7 to find a PR FIR synthesis polyphase matrix and a sampling matrix such that collected data is the minimum among all sampling matrices up to equivalence. We also provide Algorithm 9 to find a PR FIR synthesis polyphase matrix where delays in each analysis filters are allowed. Once having a particular synthesis matrix, we can parametrize the set of all synthesis matrices where an optimal solution can be found for given design criteria.

Chapter 6

Density on the Left Invertible Set or Left NonInvertible Set

In this section, We will show the set of left invertible $N \times P$ matrices over a polynomial ring $\mathbb{C}[z_1, .., z_M]$ is dense if $N - P \geq M$, while the set of left non-invertible $N \times P$ matrices over a polynomial ring $\mathbb{C}[z_1, .., z_M]$ is dense for all $N, P, M \geq 1$.

Definition 33. *A set $\mathcal{S}$ in the power set of polynomial matrices $\mathcal{P}(\mathbb{C}[z_1, ..., z_M]^{N\times P})$ is* dense *if given any $H(\boldsymbol{z}) \in \mathbb{C}[z_1, ..., z_M]^{N\times P}$ and for all $\epsilon > 0$, there exists*

$$H^{(\epsilon)}(\boldsymbol{z}) = \sum_{\alpha_1 \dots \alpha_M} h^{(\epsilon)}_{ij\alpha_1\dots\alpha_M} z_1^{\alpha_1} \dots z_M^{\alpha_M}$$

in $\mathcal{S}$ with $|h^{(\epsilon)}_{ij\alpha_1\dots\alpha_M} - h_{ij\alpha_1\dots\alpha_M}| < \epsilon$ for all $i, j, \alpha_1, ..., \alpha_M$.

6.1 Density on the Left Invertible Set over a Polynomial Ring

Our main objective is to prove the following Theorem.

Theorem 17. *For $N - P \geq M$, a set $\mathcal{I}^{\infty}_{N,P,M}$ of $N \times P$ left invertible matrices over $\mathbb{C}[z_1, ..., z_M]$ is dense.*

Proposition 9. *In $\mathbb{C}^M$, for any given set $\{H_i(\boldsymbol{z})\}_{i=1}^M$, for all ϵ, there exists M-variate polynomial matrix $(H_1^{(\epsilon)}(\boldsymbol{z}), ..., H_M^{(\epsilon)}(\boldsymbol{z}))$ with $|h^{(\epsilon)}_{i\alpha_1\dots\alpha_M} - h_{i\alpha_1\dots\alpha_M}| < \epsilon$ such that $\{H_i^{(\epsilon)}(\boldsymbol{z})\}_{i=1}^M$ is Gröbner basis of*

$$\langle H_1^{(\epsilon)}(\boldsymbol{z}), ..., H_M^{(\epsilon)}(\boldsymbol{z})\rangle$$

with respect to degree lexicographical order.

Proof. According to Lemma 1 and Theorem 1, it is enough to show there exists M-variate polynomial matrix

$$(H_1^{(\epsilon)}(\boldsymbol{z}), ..., H_M^{(\epsilon)}(\boldsymbol{z}))$$

with $|h^{(\epsilon)}_{i\alpha_1\dots\alpha_M} - h_{i\alpha_1\dots\alpha_M}| < \epsilon$ such that for all $i \neq j$, the leading powers $lp(H_i^{(\epsilon)}(\boldsymbol{z})/d)$ and $lp(H_j^{(\epsilon)}(\boldsymbol{z})/d)$

are relatively prime. We may assume the leading powers $lp(H_1(\boldsymbol{z})) \geq lp(H_2(\boldsymbol{z})) \geq ... \geq lp(H_M(\boldsymbol{z}))$. Let $z_1^{\beta_1}...z_M^{\beta_M} = lp(H_1(\boldsymbol{z}))$. Let $n = \sum_{i=1}^M \beta_i + 1$. Choose $|\epsilon_1| < \epsilon$, $H_1^{(\epsilon)}(\boldsymbol{z}) = \epsilon_1 z_1^n + H_1(\boldsymbol{z})$. Choose $|\epsilon_2| < \epsilon$, $H_2^{(\epsilon)}(\boldsymbol{z}) = \epsilon_2 z_2^n + H_2(\boldsymbol{z})$ such that $\gcd(H_1^{(\epsilon)}(\boldsymbol{z}), H_2^{(\epsilon)}(\boldsymbol{z})) = 1$. To find ϵ_2 such that $\gcd(H_1^{(\epsilon)}(\boldsymbol{z}), H_2^{(\epsilon)}(\boldsymbol{z})) = 1$, firstly we factorize into irreducible polynomial $H_1^{(\epsilon)}(\boldsymbol{z}) = f_1 f_2 ... f_k$. Then we find ϵ_2 such that none of f_i divides $H_2^{(\epsilon)}(\boldsymbol{z})$. Now choose $|\epsilon_3| < \epsilon$, $H_3^{(\epsilon)}(\boldsymbol{z}) = \epsilon_3 z_3^n + H_3(\boldsymbol{z})$ such that $\gcd(H_i^{(\epsilon)}(\boldsymbol{z}), H_3^{(\epsilon)}(\boldsymbol{z})) = 1, i = 1, 2$.

....

Choose $|\epsilon_M| < \epsilon$, $H_M^{(\epsilon)}(\boldsymbol{z}) = \epsilon_M z_M^n + H_M(\boldsymbol{z})$ such that $\gcd(H_i^{(\epsilon)}(\boldsymbol{z}), H_M^{(\epsilon)}(\boldsymbol{z})) = 1$ for $i < M$. Thus for all $i \neq j$, the leading powers $lp(H_i^{(\epsilon)}(\boldsymbol{z})/d)$ and $lp(H_j^{(\epsilon)}(\boldsymbol{z})/d)$ are relatively prime, where

$$d = \gcd(H_i^{(\epsilon)}(\boldsymbol{z}), H_j^{(\epsilon)}(\boldsymbol{z})) = 1$$

for $i \neq j$. Therefore $\{H_i^{(\epsilon)}(\boldsymbol{z})\}_{i=1}^M$ is Gröbner basis of $\langle H_1^{(\epsilon)}(\boldsymbol{z}), ..., H_N^{(\epsilon)}(\boldsymbol{z})\rangle$ with respect to degree lexicographical order. □

Proposition 10. *In $\mathbb{C}^M$, $V(\{H_i(\boldsymbol{z})\}_{i=1}^M) \neq \emptyset$, there exists $H^{(\epsilon)}(\boldsymbol{z})$, where $|h_{i\alpha_1...\alpha_M}^{(\epsilon)} - h_{i\alpha_1...\alpha_M}| < \epsilon$ for all $i, \alpha_1, ..., \alpha_M$ such that $V(\{H_i^{(\epsilon)}(\boldsymbol{z})\}_{i=1}^M)$ is finite set.*

Proof. Let $z_1^{\beta_1}...z_M^{\beta_M} = lp(H_1(\boldsymbol{z}))$. Let $n = \sum_{i=1}^M \beta_i + 1$. By Theorem 3, we construct $\{H_i^{(\epsilon)}(\boldsymbol{z})\}_{i=1}^M$ from $\{H_i(\boldsymbol{z})\}_{i=1}^M$, where $H_i^{(\epsilon)}(\boldsymbol{z}) = \epsilon_1 z_i^n + H_i(\boldsymbol{z})$ with $|\epsilon_i| < \epsilon$ such that

$$\gcd(H_i^{(\epsilon)}(\boldsymbol{z}), H_j^{(\epsilon)}(\boldsymbol{z})) = 1$$

for $i \neq j$. Since for all $i = 1, ..., n$, $lp(H_i^{(\epsilon)}(\boldsymbol{z})) = z_i^n$. Then by Theorem 3, this implies that $V(\{H_i^{(\epsilon)}(\boldsymbol{z})\}_{i=1}^M)$ is finite set. □

Proposition 11. *[14] Suppose $H(\boldsymbol{z}) = (H_1(\boldsymbol{z}), ..., H_N(\boldsymbol{z}))^T$ is an $N \times 1$ a polynomial matrix. Let $\mathcal{H} = \{H_1(\boldsymbol{z}), ..., H_N(\boldsymbol{z})\}$. TFAE:*

(1) $H(\boldsymbol{z})$ is polynomial left invertible;

(2) The variety $V(\mathcal{H}) = \emptyset$;

Proposition 12. *[61, p.92] Suppose $H(\boldsymbol{z}) = (H_1(\boldsymbol{z}), ..., H_N(\boldsymbol{z}))^T$ is an $N \times 1$ a polynomial matrix. Let $\mathcal{H} = \{H_1(\boldsymbol{z}), ..., H_N(\boldsymbol{z})\}$. TFAE:*

(1) $H(\boldsymbol{z})$ is polynomial left invertible;

(2) The reduced Gröbner basis of $\mathcal{H}$ is $\{1\}$.

Proposition 13. *[42] A is a $N \times P$ M-variate polynomial matrix, $N \geq P$. The following are equivalent:*

(1) *A is polynomial left invertible;*

(2) *A can be completed to a square $N \times N$ polynomial left invertible matrix $\overline{A}$;*

(3) *There exists a polynomial left invertible matrix E such that $EA = \begin{bmatrix} I_P \\ 0 \end{bmatrix}$.*

Lemma 13. *For $N-1 = M$, a set of $N \times 1$ left invertible matrices over $\mathbb{C}[z_1, ..., z_M]$ is dense.*

Proof. Now $N = M+1$. Given $H(\boldsymbol{z}) \in \mathbb{C}[z_1, ..., z_M]^{(M+1)\times 1}$ and some $\epsilon > 0$. If $H(\boldsymbol{z})$ is polynomial left invertible, then we have nothing to prove. Suppose $H(\boldsymbol{z})$ is not polynomial left invertible. By Theorem 11, $V(\{H_i(\boldsymbol{z})\}_{i=1}^{M+1}) \neq \emptyset$. This implies $V(\{H_i(\boldsymbol{z})\}_{i=1}^{M}) \neq \emptyset$. Then by Proposition 10, there exists $H^{(\epsilon)}(\boldsymbol{z})$, where $|h^{(\epsilon)}_{i\alpha_1...\alpha_M} - h_{i\alpha_1...\alpha_M}| < \epsilon$ for all $i, \alpha_1, ..., \alpha_M$ such that

$$V(\{H_i^{(\epsilon)}(\boldsymbol{z})\}_{i=1}^{M})$$

is finite set. Let $V(\{H_i^{(\epsilon)}(\boldsymbol{z})\}_{i=1}^{M}) = \{P_j\}_{j=1}^{l}$. Suppose for all $j \leq l$, $H_{M+1}(P_j) \neq 0$. Then done. If $H_{M+1}(P_{j_1}) = 0$, we can choose $\delta < \epsilon/l$, set $H^{(1)}_{M+1}(\boldsymbol{z}) = H_{M+1}(\boldsymbol{z}) + \delta$. Then $H^{(1)}_{M+1}(P_{i_1}) \neq 0$. If $H_{M+1}(P_{j_2}) = 0$, set $H^{(2)}_{M+1}(\boldsymbol{z}) = H_{M+1}(\boldsymbol{z}) + 2\delta$. Then $H^{(2)}_{M+1}(P_{j_i}) \neq 0$ for $i = 1, 2$. We can construct that repeatedly until there exists $k \leq l$, $H^{(k)}_{M+1}(\boldsymbol{z}) = H_{M+1}(\boldsymbol{z}) + k\delta$ such that $H^{k}_{M+1}(P_{j_r}) \neq 0$ for $r \geq k$. Also we know that$H^{(k)}_{M+1}(P_{j_r}) \neq 0$ for $r < k$. Then set $H^{(\epsilon)}_{M+1}(\boldsymbol{z}) = H^{(k)}_{M+1}(\boldsymbol{z})$. Therefore $V(\{H_i^{(\epsilon)}(\boldsymbol{z})\}_{i=1}^{M+1}) = \emptyset$. Then by Theorem 11, $H^{(\epsilon)}(\boldsymbol{z})$ is polynomial left invertible. Therefore a set of $N \times 1$ left invertible matrices over $\mathbb{C}[z_1, ..., z_M]$ is dense. □

Notation 2. *We denote $A \sim B$ if there exists a square invertible matrix E such that $A = EB$.*

Proposition 14. *For $N-1 \geq M$, a set $\mathcal{I}^{\infty}_{N,1,M}$ of $N \times 1$ left invertible matrices over $\mathbb{C}[z_1, ..., z_M]$ is dense.*

Proof. Simply from the fact that $V(\{H_i^{(\epsilon)}(\boldsymbol{z})\}_{i=1}^{N}) \subset V(\{H_i^{(\epsilon)}(\boldsymbol{z})\}_{i=1}^{M+1}) = \emptyset$ as $M+1 \leq N$. By Theorem 11, a set of $N \times 1$ left invertible matrices over $\mathbb{C}[z_1, ..., z_M]$ is dense. □

Now let us consider when $P > 1$.

Proof. of Theorem 17. When $P = 1$, it has been done. Assume the statement is true for any $N \times (P-1)$ matrix such that $N - (P-1) \geq M$. Let $H(\boldsymbol{z})$ is $N \times P$ matrix such that $N - P \geq M$. Let $H'(\boldsymbol{z})$ be $N \times (P-1)$ sub-matrix of $H(\boldsymbol{z})$. i.e. $H(\boldsymbol{z}) = (H'(\boldsymbol{z})\, \boldsymbol{a}(\boldsymbol{z}))$ where $\boldsymbol{a}(\boldsymbol{z}) = (H_{1P}(\boldsymbol{z}), ..., H_{NP}(\boldsymbol{z}))^T$. Since $N - (P-1) > N - P \geq M$, so all $\epsilon > 0$, there exists M-variate polynomial matrix $H'^{(\epsilon)}(\boldsymbol{z})$, where $|h^{(\epsilon)}_{i\alpha_1...\alpha_M} - h_{i\alpha_1...\alpha_M}| < \epsilon$ for all $i, \alpha_1, ..., \alpha_M$ such that $H'^{(\epsilon)}(\boldsymbol{z})$ is polynomial left invertible. By Proposition

13, there exists polynomial left invertible $E'(\boldsymbol{z})$ such that $E'(\boldsymbol{z})H'^{(\epsilon)}(\boldsymbol{z}) = \binom{I_{P-1}}{0}$. Let $\boldsymbol{b}(\boldsymbol{z}) = E'(\boldsymbol{z})\boldsymbol{a}(\boldsymbol{z})$. Let $E'^{-1} = (d_{ij})$. Let $\gamma = \max\{\text{coefficient of } d_{ij}\}_{1\leq i,j\leq N}$. Now consider the subset $\{b_{P-1+i}\}_{i=1}^{M+1}$ of $\{b_i\}_{i=1}^{N}$. We may assume that the leading powers $lp(b_P(\boldsymbol{z})) \geq lp(b_{P+1}(\boldsymbol{z})) \geq ... \geq lp(b_{P+M}(\boldsymbol{z}))$. Let $z_1^{\beta_1}...z_M^{\beta_M} = lp(b_P(\boldsymbol{z}))$. Let $n = \sum_{i=1}^{M}\beta_i + 1$. Since size of $\{b_{P-1+i}(\boldsymbol{z})\}_{i=1}^{M+1}$ is $M+1 \geq M$, by the based case, we can construct $\{b_{P-1+i}^{(\epsilon)}(\boldsymbol{z})\}_{i=1}^{M+1}$ from $\{b_{P-1+i}(\boldsymbol{z})\}_{i=1}^{M+1}$, where $b_{P-1+i}^{(\epsilon)}(\boldsymbol{z}) = \epsilon_i z_i^n + b_{P-1+i}(\boldsymbol{z})$ for $1 \leq i \leq M$ with $|\epsilon_i| < \frac{\epsilon}{\gamma(M+1)}$ such that

$$\gcd(b_{P-1+i}^{(\epsilon)}(\boldsymbol{z}), b_{P-1+j}^{(\epsilon)}(\boldsymbol{z})) = 1$$

for $1 \leq i, j \leq M, i \neq j$ and $b_{P+M}^{(\epsilon)}(\boldsymbol{z}) = b_{P+M}(\boldsymbol{z}) + \epsilon_{M+1}$ with $|\epsilon_{M+1}| < \frac{\epsilon}{\gamma(M+1)}$ such that

$$V(\{b_{P-1+i}^{(\epsilon)}(\boldsymbol{z})\}_{i=1}^{M+1}) = \emptyset.$$

Thus the reduced Gröbner basis of $\{b_{P-1+i}^{(\epsilon)}(\boldsymbol{z})\}_{i=1}^{M+1}$ is $\{1\}$. Then by Theorem 12 and proposition 13, $(b_{P-1+i}^{(\epsilon)}(\boldsymbol{z}), ..., b_N^{(\epsilon)}(\boldsymbol{z}))^T \sim (1, 0, ..., 0)^T$. Now let

$$\boldsymbol{a}^{(\epsilon)}(\boldsymbol{z}) = (E'(\boldsymbol{z}))^{-1}(b_1(\boldsymbol{z}), ..., b_{P-1}(\boldsymbol{z}), b_P^{(\epsilon)}(\boldsymbol{z}), ..., b_{P+M}^{(\epsilon)}(\boldsymbol{z}), b_{P+M+1}(\boldsymbol{z}), ..., b_N(\boldsymbol{z}))^T$$

$$= (H_{1P}(\boldsymbol{z}) + \sum_{j=1}^{M+1} d_{1(P-1+j)}\epsilon_j z_j^n, ..., H_{NP}(\boldsymbol{z}) + \sum_{j=1}^{M+1} d_{N(P-1+j)}\epsilon_j z_j^n)^T.$$

Now let $H^{(\epsilon)}(\boldsymbol{z}) = (H'(\boldsymbol{z})\,\boldsymbol{a}^{(\epsilon)}(\boldsymbol{z}))$. Then for each coefficient $h_{iP\alpha}^{(\epsilon)}$ of $H_{iP}^{(\epsilon)}(\boldsymbol{z})$, $|h_{iP\alpha}^{(\epsilon)} - h_{iP\alpha}| < |(M+1)\gamma\frac{\epsilon}{\gamma(M+1)}| = \epsilon, i = 1, ...N$. Also

$$H^{(\epsilon)}(\boldsymbol{z}) \sim \begin{pmatrix} 1 & 0 & \cdots & 0 & b_1(\boldsymbol{z}) \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & b_{P-1}(\boldsymbol{z}) \\ 0 & \cdots & \cdots & 0 & b_P^{(\epsilon)}(\boldsymbol{z}) \\ \vdots & \cdots & \cdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & b_{P+M}^{(\epsilon)}(\boldsymbol{z}) \\ 0 & \cdots & \cdots & 0 & b_{P+M+1}(\boldsymbol{z}) \\ \vdots & \cdots & \cdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & b_N(\boldsymbol{z}) \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & \cdots & 0 & b_1(\boldsymbol{z}) \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & b_{P-1}(\boldsymbol{z}) \\ 0 & \cdots & \cdots & 0 & 1 \\ \vdots & \cdots & \cdots & \vdots & 0 \\ \vdots & \cdots & \cdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 \\ 0 & \cdots & \cdots & 0 & b_{P+M+1}(\boldsymbol{z}) \\ \vdots & \cdots & \cdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & b_N(\boldsymbol{z}) \end{pmatrix}$$

By using row operations, we have

$$H^{(\epsilon)}(\boldsymbol{z}) \sim \begin{pmatrix} I_P \\ 0 \end{pmatrix}.$$

By Proposition 13, $\mathrm{H}^{(\epsilon)}(\boldsymbol{z})$ is polynomial left invertible. Therefore a set of $N \times P$ left invertible matrices over $\mathbb{C}[z_1, ..., z_M]$ is dense for $N - P \geq M$. □

6.2 Density on the Left NonInvertible Set over a Polynomial Ring

Lemma 14. *The set $\mathcal{N}_{N,P,0}$ of left noninvertible $N \times P$ matrices over $\mathbb{C}$ is closed.*

Proof. Let $\mathcal{T}$ be the set of $N \times P$ matrices over $\mathbb{C}$. First we know if $P > N$, then every matrix is left noninvertible. Since $\mathcal{T}$ is closed, the set of $N \times P$ matrices over $\mathbb{C}$ is closed if $P > N$. For $N \geq P$, define $f : \mathcal{T} \to \mathbb{C}^{\binom{N}{P}}$ be given by taking the maximal minors of a matrix using some order. Obviously f is continuous map. It is enough to show that every left noninvertible matrix maps to $(0, 0, ..., 0)$ and every left invertible matrix does not map to $(0, 0, ..., 0)$. Since $\{(0, 0, ..., 0)\}$ is closed in $\mathbb{C}^{C_P^N}$ and f continuous, this implies $\mathcal{N}_{N,P,0}$ is closed too. Now we want to show that every left invertible matrix does not map to $(0, 0, ..., 0)$ and every left noninvertible matrix maps to $(0, 0, ..., 0)$. Suppose the maximal minors of T are all zero. This implies $T \not\sim \binom{I_P}{0}$. Thus T is not invertible. Therefore every left invertible matrix does not map to $(0, 0, ..., 0)$. Suppose T is a left noninvertible matrix, which does not map to $(0, 0, ..., 0)$. Let

$\{d_i\}_{i=1}^{C_P^N}$ be the set of maximal minors of T. Since d_i are not all zero, there exist $\{e_i\}_{i=1}^{C_P^N}$ such that

$$\sum_{i=1}^{C_P^N} e_i d_i = 1.$$

Let $\{T_i\}$ are the set of $P \times P$ submatrices of T corresponding to the set of maximal minors of T. By cramer's rule,

$$\text{adj}(T_i)T_i = d_i I.$$

Let G_i be the $P \times N$ matrix extended from $\text{adj}(T_i)$ by adding zero columns corresponding to those rows in T which are not in T_i. Then

$$G_i H_i = d_i I.$$

Let

$$G = \sum_{i=1}^{C_P^N} (e_i G_i).$$

So

$$GH = \sum_{i=1}^{C_P^N} (e_i G_i) H = \sum_{i=1}^{C_P^N} e_i d_i I = I$$

which contradicts that H is left noninvertible. Therefore every left noninvertible matrix maps to $(0, 0, ..., 0)$. From these, we concludes that $\mathcal{N}_{N,P,0}$ is closed. □

Lemma 15. *The set $\mathcal{N}_{N,P,0}$ of left noninvertible $N \times P$ matrices over $\mathbb{C}$ is not dense for $N \geq P$.*

Proof. Suppose the set $\mathcal{N}_{N,P,0}$ of left noninvertible $N \times P$ matrices over $\mathbb{C}$ is dense. Since $\mathcal{N}_{N,P,0}$ is closed, $\mathcal{N}_{N,P,0} = \mathbb{C}^{N \times P}$, which contradicts as $\binom{I_P}{0}$ is left invertible $N \times P$ matrices over $\mathbb{C}$. □

Lemma 16. *The set $\mathcal{I}_{N,P,0}$ of left invertible $N \times P$ matrices over $\mathbb{C}$ is dense for $N \geq P$.*

Proof. Define $f : \mathcal{T} \to \mathbb{C}^{\binom{N}{P}}$ be given by taking the maximal minors of a matrix using some order. Obviously f is an non-constant analytic map. By Open Mapping Theorem, the set of left noninvertible does not contain any open set. Thus $\mathcal{I}_{N,P,0}$ is a dense set. □

Proposition 15. *The set of left noninvertible $N \times P$ matrices over $\mathbb{C}[z_1, ...z_M]$ is dense for $M > 0$.*

Proof. To show the set of left noninvertible $N \times P$ matrices over $\mathbb{C}[z_1, ...z_M]$ is dense for $M > 0$, it is enough to show given $H(\boldsymbol{z})$ be an left invertible $N \times P$ matrices over $\mathbb{C}[z_1, ...z_M]$, given any $\epsilon > 0$, there exist $N \times P$ matrix $E(\boldsymbol{z})$ over $\mathbb{C}[z_1, ...z_M]$, where the absolute value of coefficients of polynomials is less than ϵ, such that $(E + H)(\boldsymbol{z})$ is left noninvertible over $\mathbb{C}[z_1, ...z_M]$.

Let $H(\boldsymbol{z})$ be an left invertible $N \times P$ matrices over $\mathbb{C}[z_1, ...z_M]$. Let $(H_{11}(\boldsymbol{z}), H_{21}(\boldsymbol{z}), ..., H_{N1}(\boldsymbol{z}))^T$ be the first column of $H(\boldsymbol{z})$. Let $z_1 = \delta$ for some $\delta > 1$, $z_i = 1$ for $i = 2, ..., M$. Given $\epsilon > 0$, there exist $r_1, ...r_M$ such that

$$\epsilon_1 \delta^{r_1} + H_{11}(\delta, 1, ..., 1) = 0$$

$$\epsilon_2 \delta^{r_2} + H_{21}(\delta, 1, ..., 1) = 0$$

$$......................................$$

$$\epsilon_N \delta^{r_N} + H_{N1}(\delta, 1, ..., 1) = 0$$

where $|\epsilon_i| < \epsilon$ for all i. Let $E_{i1}(\boldsymbol{z}) = \epsilon_i z_1^{r_1} + H_{i1}(\boldsymbol{z})$ for $i = 1, ..., N$, and $E_{ij}(\boldsymbol{z}) = 0$ for $j \neq 1$. So

$$(E+H)(\delta, 1, ..., 1) = \begin{pmatrix} 0 & H_{12}(\delta, 1, ..., 1) & \cdots & H_{1P}(\delta, 1, ..., 1) \\ 0 & H_{22}(\delta, 1, ..., 1) & \cdots & H_{2P}(\delta, 1, ..., 1) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & H_{N2}(\delta, 1, ..., 1) & \cdots & H_{NP}(\delta, 1, ..., 1) \end{pmatrix}$$

Thus $(E+H)(\boldsymbol{z})$ is left noninvertible over $\mathbb{C}[z_1, ...z_M]$. Therefore the set of left noninvertible $N \times P$ matrices over $\mathbb{C}[z_1, ...z_M]$ is dense for $M > 0$. ☐

When $N - P < M$, we do not know the set of left invertible $N \times P$ matrices over $\mathbb{C}[z_1, ...z_M]$ is dense or not dense in general.

Lemma 17. *For $N = P$ and $M > 0$, define $d : \mathbb{C}[z_1, ...z_M]^{N \times N} \to \mathbb{C}[z_1, ...z_M]$ to be a determinant map. If $H(\boldsymbol{z})$ is polynomial left invertible, then the $d(H(\boldsymbol{z}))$ is a non-zero constant in $\mathbb{C}[z_1, ...z_M]$ (Note: non-zero constant in $\mathbb{C}[z_1, ...z_M]$ is not dense). Since $d^{-1}(d(\mathcal{I}^{\infty}_{N,N,M})) = \mathcal{I}^{\infty}_{N,N,M}$ and $d^{-1}(d(\mathcal{N}^{\infty}_{N,N,M})) = \mathcal{N}^{\infty}_{N,N,M}$ and d is continuous, the set of left invertible $N \times N$ matrices over $\mathbb{C}[z_1, ...z_M]$ is not dense.*

Proposition 16. *If $N - P \geq M$ and $k > 0$, then the set $\mathcal{N}^k_{N,P,M}$ of left noninvertible $N \times P$ polynomial matrices of degree at most k is not dense; whereas if $N - P < M$ and $k > 0$, then the set $\mathcal{I}^k_{N,P,M}$ of left invertible $N \times P$ polynomial matrices of degree at most k is not dense.*

Proof. Follows immediately by Theorem 8, Theorem 11, and Lemma 3. ☐

Proposition 17. *Same notation as above. Let k be a some fixed positive integer and let D be a compact set. Then there not exists a sequence $\{(E_i + H)(\boldsymbol{z})\}$ of polynomial left noninvertible matrices on D converges to $H(\boldsymbol{z})$ where every entry of $E_i(\boldsymbol{z})$ is a polynomial of degree k or less.*

	$N-P \geq M$	$N-P < M$
$\mathcal{N}_{N,P,0}$	Not dense	Empty set
$\mathcal{I}_{N,P,0}$	Dense	Whole set
$\mathcal{N}^{\infty}_{N,P,M}$ with $M > 0$	Dense	Dense
$\mathcal{I}^{\infty}_{N,P,M}$ with $M > 0$	Dense	Not known yet
$\mathcal{N}^{k}_{N,P,M}$ with $k < \infty$	Not dense	Dense
$\mathcal{I}^{k}_{N,P,M}$ with $k < \infty$	Dense	Not dense
$\mathcal{N}^{k}_{N,P,M,D}$	Not dense	Not dense

Table 6.1: Density on $\mathcal{N}^{k}_{N,P,M}$, $\mathcal{I}^{k}_{N,P,M}$ and $\mathcal{N}^{k}_{N,P,M,D}$

Proof. Let $H(\boldsymbol{z})$ be an left invertible $N \times P$ matrices over $\mathbb{C}[z_1, ...z_M]$. Recall $E_i(\boldsymbol{z})$ is a $N \times P$ matrix over $\mathbb{C}[z_1, ...z_M]$, where the absolute value of coefficients of polynomials is less than some ϵ_i, such that $(E_i + H)(\boldsymbol{z})$ is left noninvertible over $\mathbb{C}[z_1, ...z_M]$. Suppose a sequence $\{(E_i + H)(\boldsymbol{z})\}$ converges to $H(\boldsymbol{z})$. Since $(E_i + H)(\boldsymbol{z})$ is left noninvertible over $\mathbb{C}[z_1, ...z_M]$ on D, there exists $(z_{i1}, z_{i2}, ..., z_{iM}) \in D$ such that $(E_i + H)(z_{i1}, z_{i2}, ..., z_{iM})$ is left noninvertible. Now we have a infinite sequence $\{(z_{i1}, z_{i2}, ..., z_{iM})\}_{i=1...\infty}$. Since D is a compact set, there exist a subsequence $\{(z_{i_j1}, z_{i_j2}, ..., z_{i_jM})\}_{j=1...\infty}$ converges to $(\alpha_1, \alpha_2, ..., \alpha_M)$ for some $\alpha_i < \infty$. Now consider the sequence of polynomial left noninvertible matrix

$$\{(E_{i_j} + H)(z_{i_j1}, z_{i_j2}, ..., z_{i_jM})\}_{j=1...\infty}.$$

Since $H(\boldsymbol{z})$ and k are fixed and ϵ_i converges to zero and D is bounded, this sequence is bounded and converges to $H(\alpha_1, \alpha_2, ..., \alpha_M)$. By above Lemma 14, this implies that $H(\alpha_1, \alpha_2, ..., \alpha_M)$ is left noninvertible. i.e. $H(\boldsymbol{z})$ is left noninvertible over $\mathbb{C}[z_1, ...z_M]$, which contradicts that $H(\boldsymbol{z})$ is left invertible over $\mathbb{C}[z_1, ...z_M]$. Therefore there not exists a sequence $\{(E_i + H)(\boldsymbol{z})\}$ of polynomial left noninvertible matrices on D converges to $H(\boldsymbol{z})$ where every entry of $E_i(\boldsymbol{z})$ is a polynomial of degree k or less. □

Theorem 18. *Let $\mathcal{N}^{k}_{N,P,M,D}$ be a set of $N \times P$ polynomial left noninvertible matrix on a compact set D over $\mathbb{C}[z_1, ..., z_M]$ such that every entry is a polynomial of degree less than or equal to k where $k > 0$. Then $\mathcal{N}^{k}_{N,P,M,D}$ is not dense.*

Proof. Followed by the Proposition 16. □

6.3 Conclusion

To summarize, we have table 6.1 where $\mathcal{I}^{k}_{N,P,M}$ is a set of $N \times P$ left invertible polynomial matrices of degree at most k, $\mathcal{N}^{k}_{N,P,M}$ is a set of $N \times P$ left noninvertible polynomial matrices of degree at most k, $\mathcal{N}^{k}_{N,P,M,D}$

is the set $N \times P$ noninvertible polynomial matrices on a compact set D of degree at most k.

Chapter 7

Stability and Generalized Inverses

In this section, we will extend the stability and generalized Inverses from matrix with complex numbers to Laurent polynomial matrices.

7.1 Generalized Inverses

Definition 34. *Suppose that $A \in \mathbb{C}^{m\times n}$ and $b \in \mathbb{C}^m$. Then a vector $u \in \mathbb{C}^n$ is called a* least squares solution *to $Ax - b$ if $||Au - b||_2 \leq ||Av - b||_2$ for all $v \in \mathbb{C}^n$. A vector u is called a* minimal least squares solution *to $Ax - b$ and $||u||_2 < ||w||_2$ for all other least squares solutions w.*

Notation 3. *$R(A)$ denotes the range of A. $N(A)$ denotes the null space of A. A^* denotes the conjugate transpose of A.*

Definition 35. *If M is subspace of $\mathbb{C}^n$, then we define the* orthogonal projector*, P_M of $\mathbb{C}^n$ onto M by $P_M u = u$ if $u \in M$ and $P_M u = 0$ if $u \in M^\perp$.*

Definition 36. *If $A \in \mathbb{C}^{m\times n}$, then the* generalized inverse *of A is defined to be the unique matrix $A^\dagger$ such that*

(a) $AA^\dagger A = A$,

(b) $A^\dagger AA^\dagger = A^\dagger$,

(c) $(AA^\dagger)^* = AA^\dagger$,

(d) $(A^\dagger A)^* = A^\dagger AA$

or equivalent to

(1) $AA^\dagger = P_{R(A)}$,

(2) $A^\dagger A = P_{R(A^\dagger)}$ *where A^* is a complex conjugate of A.*

Lemma 18. *[30, p.13] If $A \in \mathbb{C}^{n\times k}$ and $B \in \mathbb{C}^{k\times m}$, then*

$$\text{rank}(AB) \leq min\{\text{rank}(A), \text{rank}(B)\}.$$

Lemma 19. *[30, p.13] If* $A \in \mathbb{C}^{n\times m}$*, then* $\text{rank}(A^*A) = \text{rank}(A)$*.*

Remark 9. *If* $A \in \mathbb{C}^{n\times n}$ *is invertible, then* $A^{-1} = A^\dagger$*.*

Proposition 18. *[9, p.12] Suppose* $A \in \mathbb{C}^{n\times m}$*,* $A^\dagger = (A^*A)^\dagger A^*$*.*

Proposition 19. *Suppose* $A \in \mathbb{C}^{n\times m}$ *is invertible. Then* $A^\dagger = (A^*A)^{-1}A^*$*.*

Proof. Follows immediately by Lemma 19, Proposition 18, and Remark 9. □

7.2 Optimization of Laurent Polynomial Invertible Matrice

Theorem 19. *[9, p.28] Suppose that* $A \in \mathbb{C}^{m\times n}$ *and* $b \in \mathbb{C}^m$*. Then* $A^\dagger b$ *is the minimal least squares solution to* $Ax = b$*.*

From above result, we know the optimal solution of $min_{G\in\mathbb{C}^{P\times N}}||G||_2$ such that $GH = I$ is $H^\dagger$.

Theorem 20. *[9, p.224] Suppose that* $A, E \in \mathbb{C}^{m\times n}$ *and* $\text{rank}(A + E) = \text{rank}(A)$*. Suppose further that* $||A^\dagger||||E|| < 1/2$ *where* $||.|| = ||.||_2$*. Let* $\alpha = ||A^\dagger||/(1 - ||A^\dagger||||P_{R(A)}EP_{R(A^*)}||)$*. Then*

$$||(A+E)^\dagger - A^\dagger|| \leq \alpha||A^\dagger||||E||(1 + (1+\alpha^2||E||^2)^{1/2})(1 + \frac{\alpha^2||E||^2}{(1-\alpha^2||E||^2)^2})^{1/2}$$

for any $||E|| < 1/\alpha^2$*.*

Now let $A(z)$ be an $N \times P$ matrix values function for $z_1, ..., z_m$. That is,

$$A(z) = [a_{ij}(z)]_{i=1...N,j=1...P}$$

Then we define the $P \times N$ matrix valued function $A^\dagger(.)$ by $A^\dagger(t) = [A(t)]^\dagger$. Similar to Theorem 19, we have following theorem.

Theorem 21. *Suppose that* $A(z)$ *be an* $N \times P$ *matrix values function for* $z_1, ..., z_m$ *and* $b(z)$ *be an* $P \times 1$ *matrix values function for* $z_1, ..., z_m$*. Then* $A^\dagger(z)b(z)$ *is the minimal least squares solution to* $A(z)x = b(z)$ *with respect to* $||.||_2$ *for all* $z \in D$*.*

From above result, we know the optimal solution of

$$\min_{G(z)\in S} ||G(z)||_2 \text{ for all } z \in D \text{ such that } G(z)A(z) = I$$

is $A^\dagger(z)$ where S is the set of $N \times P$ matrix values functions.

Similar to Theorem 20, we have the following theorem.

Theorem 22. *Suppose that* $H(z), E(z)$ *are* $N \times P$ *matrix values functions for* $z_1, ..., z_m$ *and* $\text{rank}((H+E)(z)) = \text{rank}(H(z))$. *Suppose further that for all* $z \in D$, $||H^\dagger(z)||||E(z)|| < 1/2$ *where* $||.|| = ||.||_2$. *Let* $\alpha(z) = ||H^\dagger(z)||/(1 - ||H^\dagger(z)||||P_{R(H(z))}(z)E(z)P_{R(H^*)}(z)||)$. *Then for all* $z \in D$

$$||(H+E)^\dagger(z) - H^\dagger(z)|| \leq \alpha(z)||H^\dagger(z)||||E(z)||(1 + (1+\alpha(z)^2||E(z)||^2)^{1/2})(1 + \frac{\alpha(z)^2||E(z)||^2}{(1-\alpha(z)^2||E(z)||^2)^2})^{1/2}$$

for any $||E(z)|| < 1/\alpha(z)^2$.

Definition 37. *A function* f *defined on an open subset* $D \subset \mathbb{R}^n$ *(or* $\mathbb{C}^n$ *respectively) is called* analytic *in* D *if each point* $w \in D$ *has an open neighborhood* U, $w \in U \subset D$. *such that the function* f *has a power series expansion*

$$f(z) = \sum_{v_1 \ldots v_n = 0}^{\infty} a_{v_1 \ldots v_n}(z_1 - w_1)^{v_1} \ldots (z_n - w_n)^{v_n}$$

which converges for all $z \in U$.

Theorem 23. *Suppose that* $A(z)$ *is a continuous* $m \times n$ *matrix valued function for* $z_1, ..., z_n$ *defined on* D. *Then* $A^\dagger(z)$ *is continuous on* D *if and only if* $\text{rank}(A(z))$ *is constant on* D.

Proof. same proof as [9, p.225] Campbell's Theorem 10.5.1. □

Theorem 24. *(Weierstrass Approximation Theorem) [31, p.226] If* $f : X \to \mathbb{R}$ *is a continuous function on a bounded closed subspace* X *of the Euclidean n-space* $\mathbb{R}^n$, *then, for any given positive real number* ϵ, *there exists a polynomial function* $p : \mathbb{R}^n \to R$ *(in the coordinates of* $\mathbb{R}^n$*) such that*

$$|f(x) - p(x)| < \epsilon$$

holds for every point x *of* X.

Lemma 20. *[30, p.491] Let is* A, B *are* $n \times m$ *matrix over* $\mathbb{R}$ *(or* $\mathbb{C}$*). If* $|A| \leq |B|$, *then* $||A||_2 \leq ||B||_2$.

Corollary 5. *Suppose that* $H(t) \in \mathbb{R}[t_1, ..., t_M]^{N \times P}$ *be a polynomial left invertible where* $N > P$. *Given an* $\epsilon > 0$, *then there exists a* $G(t) \in \mathbb{R}[t_1, ..., t_M]^{P \times N}$ *with* $G(t)H(t) = I$ *such that*

$$||H^\dagger(t) - G(t)||_2 < \epsilon.$$

for all $t \in D$ *where* D *is a bounded closed subset of* $\mathbb{R}^M$.

Proof. Suppose $G'(t)$ is an inverse polynomial matrix of $H(t)$. Let $S'(t) = H^\dagger(t) - G'(t)$. By Theorem 23, we know that $S'(t)$ is also continuous. Also it is easy to show

$$H^\dagger(t) = G'(t) + S'(t)(I - H(t)G'(t)).$$

Given an $\delta > 0$. Let $\theta = \frac{\delta}{N \max_{||x||_2=1} \sum_{j=1}^{P} |x_i|}$. Since x is on the unit circle, $\max_{||x||_2=1} \sum_{j=1}^{P} |x_i|$ is bounded and not zero. So $0 < \theta < \infty$. By Theorem 24, There exist $S(t)$ such that

$$\begin{aligned} ||S'(t) - S(t)||_2 &\leq ||(\theta)||_2 \text{ by Lemma 20} \\ &= \max_{||x||_2=1} ||(\theta)x||_2 \\ &\leq \theta \max_{||x||_2=1} || \sum_{j=1}^{P} |x_i| \; ||_2 \\ &\leq \theta \max_{||x||_2=1} || \sum_{j=1}^{P} |x_i| \; ||_1 \\ &= \theta N \max_{||x||_2=1} \sum_{j=1}^{P} |x_i| = \delta \end{aligned}$$

for all $t \in D$. Now let $G(t) = G'(t) + S(t)(I - H(t)G'(t))$. Given an $\epsilon > 0$, set $\delta < \frac{\epsilon}{sup_{t \in D} ||I - H(t)G'(t)||_2}$. Since D is closed and bounded , $sup_{t \in D} ||I - H(t)G'(t)||_2$ is bounded. Since $N > P$ and by Lemma 18, $||I - H(t)G'(t)||_2 \neq 0$ for all t. So $0 < \delta < \infty$. By Theorem 5, we know that $G(t)$ is an inverse of $H(t)$. So

$$\begin{aligned} ||H^\dagger(t) - G(t)||_2 &= ||(S'(t) - S(t))(I - H(t)G'(t))||_2 \\ &\leq ||S'(t) - S(t)||_2 ||I - H(t)G'(t)||_2 \\ &\leq \delta ||I - H(t)G'(t)||_2 \\ &< \epsilon \end{aligned}$$

for all $t \in D$. □

Corollary 6. *Let $||.|| = ||.||_2$ where D is bounded closed subset of $\mathbb{R}^M$. Suppose that $H(t), (H + E)(t) \in \mathbb{R}[t_1, ..., t_M]^{N \times P}$ be a polynomial left invertible. Suppose further that $||H^\dagger(t)|| ||E(t)|| < 1/2$. Given an $\epsilon > 0$, then by above corollary and Theorem 22, there exists $G(t), G'(t) \in \mathbb{R}[t_1, ..., t_M]^{P \times N}$ with $G(t)H(t) = I$ and $G'(t)(H + E)(t) = I$ such that for all $t \in D$*

$$||G(t) - G'(t)|| < \epsilon + \alpha(t)||H^\dagger(t)|| ||E(t)||(1 + (1 + \alpha(t)^2||E(t)||^2)^{1/2})(1 + \frac{\alpha(t)^2||E(t)||^2}{(1 - \alpha(t)^2||E(t)||^2)^2})^{1/2}$$

for any $||E(t)|| < 1/\alpha(t)^2$ *where* $\alpha(t) = ||H^\dagger(t)||/(1 - ||H^\dagger(t)||||P_{R(H(t))}(t)E(t)P_{R(H^*(t))}(t)||)$.

Lemma 21. *Suppose that* $H(z)$ *is left invertible for every* $z \in D$. *Then*

$$H^\dagger(z) = (H(z)^*H(z))^{-1}H(z)^*$$

on D.

Proof. By Proposition 19, done. □

Theorem 25. *Suppose that* $H(z) \in \mathbb{C}[z_1, ..., z_M]^{N\times P}$ *be left invertible on domain* D. *Then there exist* $f_{ij}, g \in \mathbb{C}[z_1, ..., z_M, z_1^*, ..., z_M^*]$ *with* $g \neq 0$ *such that* f_{ij}/g *is the* $(i, j)-$*entry of* $H^\dagger(z)$.

Proof. Then by Lemma 21, the general inverse of $H(z)$ on D is

$$H^\dagger(z) = (H(z)^*H(z))^{-1}H(z)^*.$$

Let $T = H(z)^*H(z)$. It is obvious that $T \in \mathbb{C}[z_1, ..., z_M, z_1^*, ..., z_M^*]$ and

$$\mathrm{adj}(T)T = \det(T)I.$$

and set f_{ij} be (i, j)-entry of $\mathrm{adj}(T)H(z)^*$ and set $g = \det(T)$. Then $f_{ij}, g \in \mathbb{C}[z_1, ..., z_M, z_1^*, ..., z_M^*]$ and the (i, j)-entry of $H^\dagger(z)$ is f_{ij}/g. Also since $H(z)$ is left invertible on D and by Lemma 19, it implies $g \neq 0$. □

In particularly, above theorem hold if $H(z)$ is a polynomial invertible matrix or Laurent polynomial invertible matrix.

Theorem 26. *Suppose that* $H(z) \in \mathbb{C}[z_1, ..., z_M, \frac{1}{z_1}, ..., \frac{1}{z_M}]^{N\times P}$ *be left invertible on* $\mathcal{C}$ *where* $\mathcal{C} = \{(u_1, ..., u_M) \in \mathbb{C}^M \mid |u_i| = 1$ *for* $i = 1, .., M\}$. *Then there exist* $f_{ij}, g \in \mathbb{C}[z_1, ..., z_M, z_1^*, ..., z_M^*]$ *such that* f_{ij}/g *is the* $(i, j)-$*entry of* $H^\dagger(z)$.

Proof. Let $m \in \mathbb{Z}_+^{\mathbb{M}}$ such that $z^mH(z)$ is a polynomial matrix. Obviously $z^mH(z)$ is also left invertible on $\mathcal{C}$. Then by Theorem 25, the desired result is immediate. □

Definition 38. *A function* f *is* holomorphic *on an open subset* $U \subset \mathbb{C}^n$ *if* f *are complex-differentiable at every point* $w \in U$.

In general, the generalized inverse of an left invertible $N\times P$ polynomial matrix may not be a holomorphic matrix.

Example 21. *Let* $H(z) = \begin{pmatrix} z \\ 1 \end{pmatrix}$ *be* 2×1 *matrix over* $\mathbb{C}[z]$*. It is obvious that* $H(z)$ *is left invertible over* $\mathbb{C}[z]$*. Let* $z = x + iy$*. The generalized inverse of* $H(z)$ *is*

$$H^{\dagger}(z) = \begin{pmatrix} \frac{x-iy}{x^2+y^2+1} & \frac{1}{x^2+y^2+1} \end{pmatrix}$$

As $\frac{1}{x^2+y^2+1}$ *is not* $\mathbb{C}$*-differentiable, it is not holomorphic. Thus* $H^{\dagger}(z)$ *is not a holomorphic matrix.*

Lemma 22. *[24, p.11] Let* D *is open subset of* $\mathbb{C}^n$*.* $\mathcal{O}_D$ *is a closed subring of* $\mathcal{C}_D$ *where* $\mathcal{O}_D$ *is the set of holomorphic function in* D *and* $\mathcal{C}_D$ *is the set of continuous complex valued functions in* D*.*

$A^{\dagger}(z)$ may not in $\mathbb{C}[z_1, ..., z_M, \frac{1}{z_1}, ..., \frac{1}{z_M}]^{N\times P}$.

Example 22. *Let* $H(z) = (z+1)$ *be* 1×1 *matrix over* $\mathbb{C}[z, \frac{1}{z}]$*. Then the generalized inverse of* $H(z)$ *is*

$$H^{\dagger}(z) = \begin{pmatrix} \frac{1}{z+1} \end{pmatrix}.$$

Corollary 7. *Let* $z_j = x_j + iy_j$*. Suppose an left inverse of* $H(z) \in \mathbb{C}[z_1, ..., z_M]^{N\times P}$ *is in form of* $G_1'(x,y) + iG_2'(x,y)$ *on* D *where* $G_1'(x,y), G_2'(x,y) \in \mathbb{R}[x_1, ..., x_M, y_1, ..., y_M]^{P\times N}$ *and* $N > P$*. Given an* $\epsilon > 0$*, there exists* $G_1(x,y), G_2(x,y) \in \mathbb{R}[x_1, ..., x_M, y_1, ..., y_M]^{P\times N}$ *with* $(G_1(x,y) + iG_2(x,y))H(z) = I$ *such that*

$$||H^{\dagger}(z) - (G_1(x,y) + iG_2(x,y))||_2 < \epsilon$$

for all $z \in D$ *where* D *is a bounded closed subset of* $\mathbb{C}^M$*.*

Proof. Let $||.|| = ||.||_2$. Suppose $G_1'(x,y) + iG_2'(x,y)$ is an inverse matrix of $H(z)$. Let $V'(x,y) + iU'(x,y) = H^{\dagger}(z) - (G_1'(x,y) + iG_2'(x,y))$. Then

$$H^{\dagger}(z) = (G_1'(x,y) + iG_2'(x,y)) + (V'(x,y) + iU'(x,y))(I - H(z)(G_1'(x,y) + iG_2'(x,y))).$$

Given $\epsilon > 0$. Let $\delta < \frac{\epsilon}{sup_{z\in D}||I - H(z)(G_1'(x,y) + iG_2'(x,y))||_2}$. In proof of Corollary 5, there exists $V(x,y), U(x,y)$ such that

$$||V'(x,y) - V(x,y)|| < \frac{\delta}{2}$$

and

$$||U'(x,y) - U(x,y)|| < \frac{\delta}{2}$$

on D. So

$$\begin{aligned}
&||(V'(x,y)+iU'(x,y))-(V(x,y)+iU(x,y))||\\
&\leq ||(V'(x,y)-V(x,y))+i(U'(x,y)-U(x,y))||\\
&\leq ||(V'(x,y)-V(x,y))||+||(U'(x,y)-U(x,y))||\\
&\leq \frac{\delta}{2}+\frac{\delta}{2}=\delta
\end{aligned}$$

on D. Now let $G_1(x,y)+iG_2(x,y) = (G_1'(x,y)+iG_2'(x,y))+(V(x,y)+iU(x,y))(I-H(z)(G_1'(x,y)+iG_2'(x,y)))$. Then followed by Corollary 5, we show that $G_1(x,y), G_2(x,y) \in \mathbb{R}[x_1,...,x_M,y_1,...,y_M]^{P\times N}$ with $(G_1(x,y)+iG_2(x,y))H(z)=I$ such that

$$||H^{\dagger}(z)-(G_1(x,y)+iG_2(x,y))||<\epsilon$$

on D. □

Corollary 8. *Suppose an left inverse of $H(z)\in\mathbb{C}[z_1,...,z_M]^{N\times P}$ is $G'(z)\in\mathbb{C}[z_1,...,z_M,z_1^*,...,z_M^*]^{P\times N}$ on D where $N>P$. Given an $\epsilon>0$, there exists a $G(z)\in\mathbb{C}[z_1,...,z_M,z_1^*,...,z_M^*]^{P\times N}$ with $G(z)H(z)=I$ such that*

$$||H^{\dagger}(z)-G(z)||_2<\epsilon$$

for all $z\in D$ where D is a bounded closed subset of $\mathbb{C}^M$.

Proof. Followed by above Corollary and the fact that $x_j=\frac{z_j+z_j^*}{2}$ and $y_j=\frac{z_j-z_j^*}{2i}$ for $j=1,...,M$. □

Corollary 9. *Suppose that $H(z)\in\mathbb{C}[z_1,...,z_M]^{N\times P}$ be Laurent polynomial left invertible on $\mathcal{C}$ where $N>P$. Given an $\epsilon>0$, there exists a Laurent polynomial matrix $G(z)$ with $G(z)H(z)=I$ on $\mathcal{C}$ such that*

$$||H^{\dagger}(z)-G(z)||_2<\epsilon$$

for all $z\in\mathcal{C}$ where $\mathcal{C}=\{(u_1,...,u_M)\in\mathbb{C}^M \mid |u_i|=1$ for $i=1,..,M\}$.

Proof. Since $|z_i|=1$ for $i=1,..,M$, $z_i^*=\frac{z_i^*}{z_iz_i^*}=\frac{1}{z_i}$. □

Corollary 10. *Suppose that $H(z)\in\mathbb{C}[z_1,...,z_M,\frac{1}{z_1},...,\frac{1}{z_M}]^{N\times P}$ be Laurent polynomial left invertible on $\mathcal{C}$ where $N>P$. Given an $\epsilon>0$, there exists a Laurent polynomial matrix $G(z)$ with $G(z)H(z)=I$ on $\mathcal{C}$ such that*

$$||H^{\dagger}(z)-G(z)||_2<\epsilon$$

for all $z\in\mathcal{C}$ where $\mathcal{C}=\{(u_1,...,u_M)\in\mathbb{C}^M \mid |u_i|=1$ for $i=1,..,M\}$.

Proof. Let $m \in \mathbb{Z}_+^{\mathbb{M}}$ such that $z^m H(z)$ is a polynomial matrix. Let $F(z) = z^m H(z)$ and $\delta = \inf_{z\in\mathcal{C}}\{|z^m|\}\epsilon$. Then by Corollary 9, there exists Laurent polynomial matrix $W(z)$ such that

$$||W(z) - F^{\dagger}(z)||_2 < \delta$$

on $\mathcal{C}$. Then

$$\begin{aligned} ||z^m(z^{-m}W(z) - z^{-m}F^{\dagger}(z))|| &= ||z^m(z^{-m}W(z) - H\dagger(z))||_2 \\ &= |z^m|||(z^{-m}W(z) - H\dagger(z))||_2 \\ &\geq \inf_{z\in\mathcal{C}}\{|z^m|\}||(z^{-m}W(z) - H\dagger(z))||_2 \end{aligned}$$

So

$$||(z^{-m}W(z) - H\dagger(z))||_2 < \frac{\delta}{\inf_{z\in\mathcal{C}}\{|z^m|\}} = \epsilon.$$

on $\mathcal{C}$ and $z^{-m}W(z)$ is a Laurent polynomial inverse matrix of $H(z)$. □

7.3 Conclusion

We have a short discussion on the extension of the generalized inverse to Laurent polynomial matrices. We prove that there exists a Laurent polynomial matrix $G(z)$ such that $G(z)$ is arbitrarily closed to the generalized inverse $H\dagger(z)$ in 2-norm on the unit sphere.

Chapter 8

Conclusion

8.1 Summary

In this thesis, we have studied multidimensional perfect reconstruction system theory and applications. The key property of this system is its perfect reconstruction, which ensures that an original input can be perfectly reconstructed from the output of the system.

For a multidimensional multirate system, we can represent an analysis part as $N \times P$ matrix $\boldsymbol{H}(\boldsymbol{z})$ in M-variables. In order to having a perfect reconstruction, we need a $P \times N$ matrix $\boldsymbol{G}(\boldsymbol{z})$ such that $\boldsymbol{G}(\boldsymbol{z})\boldsymbol{H}(\boldsymbol{z}) = \boldsymbol{I}$. We show that how to verify the invertibility of a matrix by using the Gröbner bases computation. Then we provide algorithm to find a particular inverse. We can then generate all inverses from a particular inverse. When designing an analysis part, one would like to know how likely the perfect reconstruction synthesis part exists. We employ the concept of "hold generically". Then we show that an $N \times P$ polynomial matrix in M-variables of degree at most k is generically polynomial (resp. : Laurent polynomial) left invertible when $N \times P \geq M$; on the other hand, $N \times P$ polynomial matrix in M-variables of degree at most k is generically polynomial (resp. : Laurent polynomial) left noninvertible when $N \times P < M$. Based on this result, we give some applications including a fast algorithm of finding a particular inverse.

Suppose we are given a system with a set of analysis filters. We propose an algorithm of finding perfect reconstruction synthesis filters and a sampling matrix, whose the collected data attain the minimum data among all sampling matrices up to equivalence. Among all inverse, we can find an optimal inverse by minimizing different norms for a given support.

Finally, we study the density of the set of invertible (resp. noninvertible) matrices. And we extend our study on the generalized inverse over Laurent polynomial matrices.

8.2 Future Work

Due to the Theorems of generic invertibility, we know that an $N \times P$ polynomial matrix in M variables is generic invertible when $N - P \geq M$. However, we still do not know how to measure the closeness to singularity. Can we extend the concept of condition number to Laurent polynomial matrices?

Instead of restricting our focus on invertible matrices, we can relax to noninvertible matrices. Therefore, another question we want to raise is that suppose $N \times P$ matrix $\boldsymbol{H}(\boldsymbol{z})$ in M variables (not necessarily invertible) and for some $\epsilon > 0$, can we solve the optimization problem, which is

$$\min_{\boldsymbol{J}(\boldsymbol{z})} \left\{ \int_{[-\pi,\pi]^M} \operatorname{tr}\left(\boldsymbol{J}(e^{j\boldsymbol{w}}) \boldsymbol{J}(e^{j\boldsymbol{w}})^H \right) d\boldsymbol{w} \right\}$$

such that $|\boldsymbol{J}(\boldsymbol{z})\boldsymbol{H}(\boldsymbol{z}) - I| < \epsilon$?

In Section 5.4 of Chapter 5, we found that an optimal inverse with respect to $\| \cdot \|_E$ or $\| \cdot \|_1$. But these inverses may have large sets of nonzero coefficients. This would be difficult to implement in practice. Therefore, we would like to investigate a way to have an optimal inverse by minimizing the norms and at the same time minimizing the numbers of nonzero coefficients.

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