A Friendly Guide to the Frame Theory and Its Application to Signal Processing

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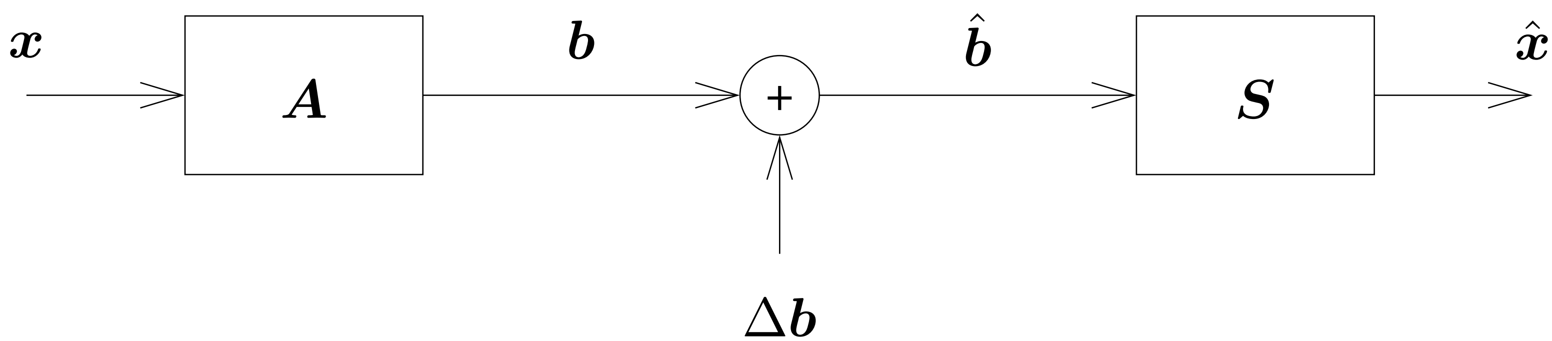
A Basic Problem

Consider the following linear inverse problem:

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ is fixed, $\boldsymbol{b} \in \mathbb{R}^m$ is given, and $\boldsymbol{x} \in \mathbb{R}^n$ is unknown.

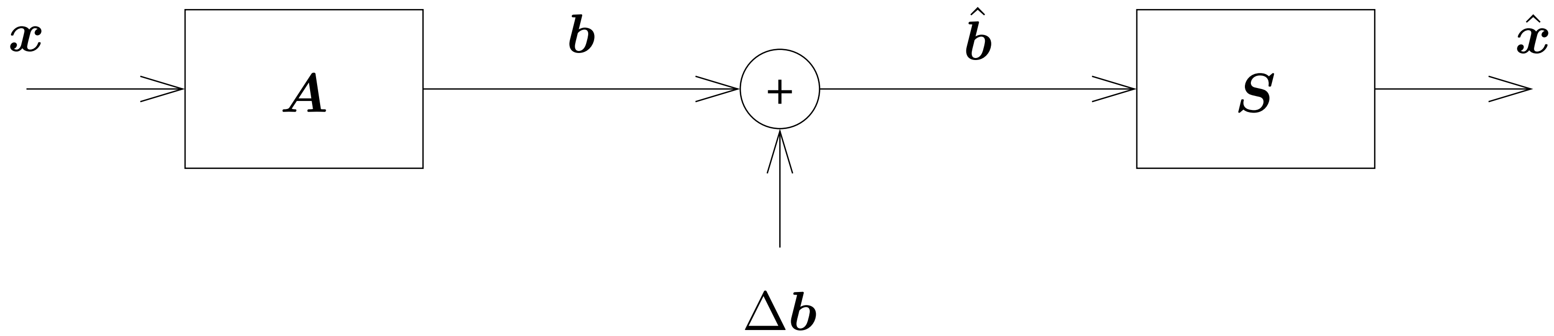


Examples: Deconvolution, computerized tomography, transform coding,...

Possible noise due to: model mismatch, measurement and/or transmission error, quantization, thresholding,...

Two Basic Questions

$$\boldsymbol{Ax} = \boldsymbol{b}$$



We have two questions:

1. Can we reconstruct $\boldsymbol{x}$ in a numerically stable way from $\boldsymbol{b}$?

2. Which is the "optimal" reconstruction algorithm in the presence of noise?

Linear Inverse Problem $Ax = b$: First Question

Question: Can we reconstruct $\boldsymbol{x}$ in a numerically stable way from $\boldsymbol{b}$?

Answer: It depends on the condition number of $\boldsymbol{A}$:

$$\kappa(\boldsymbol{A}) = \frac{\sigma_1(\boldsymbol{A})}{\sigma_n(\boldsymbol{A})}.$$

The smaller $\kappa(\boldsymbol{A})$ ($\kappa(\boldsymbol{A}) \geq 1$), the more stable (or well-conditioned) the problem is.

Intuition: σ_1 and σ_n are the largest and smallest singular values of $\boldsymbol{A}$. Thus, for all $\boldsymbol{x}$:

$$\sigma_n \|\boldsymbol{x}\|_2 \leq \|\boldsymbol{A}\boldsymbol{x}\|_2 \leq \sigma_1 \|\boldsymbol{x}\|_2$$

That means $\boldsymbol{A}$ should behavior modestly with respect to 2-norm!

Linear Inverse Problem $Ax = b$: Second Question

Question: When the system $\boldsymbol{A}$ is overcomplete, there are many (infinite) ways to reconstruct $\boldsymbol{x}$ from $\boldsymbol{b}$. Which one is "optimal"?

Answer: Use the pseudo-inverse $\boldsymbol{A}^{\dagger} = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T$

$$\hat{\boldsymbol{x}} = \boldsymbol{A}^{\dagger}\boldsymbol{b}$$

Special properties of the pseudo-inverse:

- $\boldsymbol{A}^{\dagger}$ provides the least-squares solution
  $\Rightarrow$ Eliminates the influence of errors orthogonal to the range of $\boldsymbol{A}$.

- $\boldsymbol{A}^{\dagger}$ has a minimum spectral norm among all left inverse of $\boldsymbol{A}$
  $\Rightarrow$ Recovers $\boldsymbol{x}$ and but doesn't blow up the noise.

Frames: Generalize to Hilbert Spaces

Consider $\boldsymbol{A}$ as a linear operator, $\boldsymbol{A} : \mathbb{R}^n \rightarrow \mathbb{R}^m$, then

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \quad \Longleftrightarrow \quad \boldsymbol{b}_i = \langle \boldsymbol{x}, \boldsymbol{a}_i \rangle, \quad i = 1, 2, \ldots, m$$

where $\boldsymbol{a}_i^T$ are the rows of $\boldsymbol{A}$.

Def: A sequence $\{\phi_k\}_{k \in \Gamma}$ in a Hilbert space H is a frame if there exist two constants (frame bounds) $\alpha > 0$ and $\beta < \infty$ such that for any $x \in H$

$$\alpha \|x\|^2 \leq \sum_{k \in \Gamma} |\langle x, \phi_k \rangle|^2 \leq \beta \|x\|^2.$$

Best case: $\alpha = \beta \Longrightarrow$ tight frame

Significance: $\{\phi_k\}_{k \in \Gamma}$ is a frame $\Longleftrightarrow$ one can recover $x \in H$ from $\{\langle x, \phi_k \rangle\}_{k \in \Gamma}$.

Dual Frame

Frame operator: $A : H \to l_2(\Gamma)$

$$(Ax)_k = \langle x, \phi_k \rangle, \quad \text{for } k \in \Gamma.$$

Pseudo inverse: $A^\dagger = (A^*A)^{-1}A^*$ exists and bounded because $\{\phi_k\}_{k\in\Gamma}$ is a frame.

Result: Reconstruction using pseudo inverse is related to a dual frame

$$x = A^\dagger A x = \sum_{k\in\Gamma} \langle x, \phi_k \rangle \tilde{\phi}_k$$

where the dual frame is defined as $\tilde{\phi}_k = (A^*A)^{-1}\phi_k$.

Easiest case: Tight frame ($\alpha = \beta$), $\tilde{\phi}_k = \alpha^{-1}\phi_k$.

Iterative Frame Reconstruction Algorithm

Both pseudo-inverse and dual frame computations need the inversion of A^*A, where

$$A^*Ax = \sum_{k\in\Gamma} \langle x, \phi_k \rangle \phi_k$$

Consider $R = I - \frac{2}{\alpha+\beta} A^*A$, then because $\alpha \cdot I \leq A^*A \leq \beta \cdot I$

$$\|R\| \leq \frac{\beta - \alpha}{\beta + \alpha} \leq 1.$$

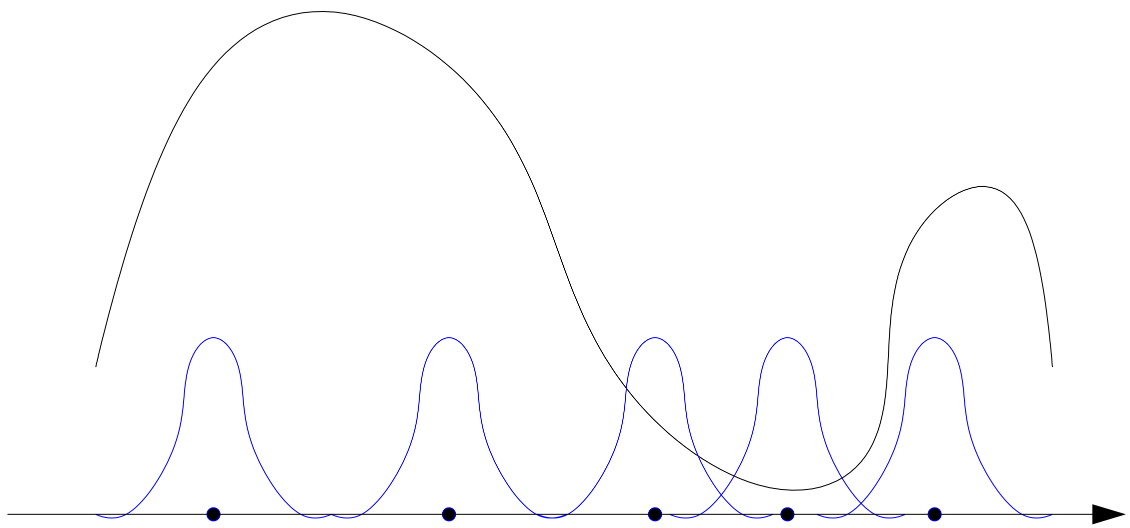
Thus,

$$(A^*A)^{-1} = \frac{2}{\alpha+\beta}(I - R)^{-1} = \frac{2}{\alpha+\beta} \sum_{i=0}^{\infty} R^i$$

Iterative reconstruction:

$$x_n = x_{n-1} + \frac{2}{\alpha+\beta} \sum_{k\in\Gamma} (\langle x, \phi_k \rangle - \langle x_{n-1}, \phi_k \rangle) \phi_k$$

Application to Generalized Sampling

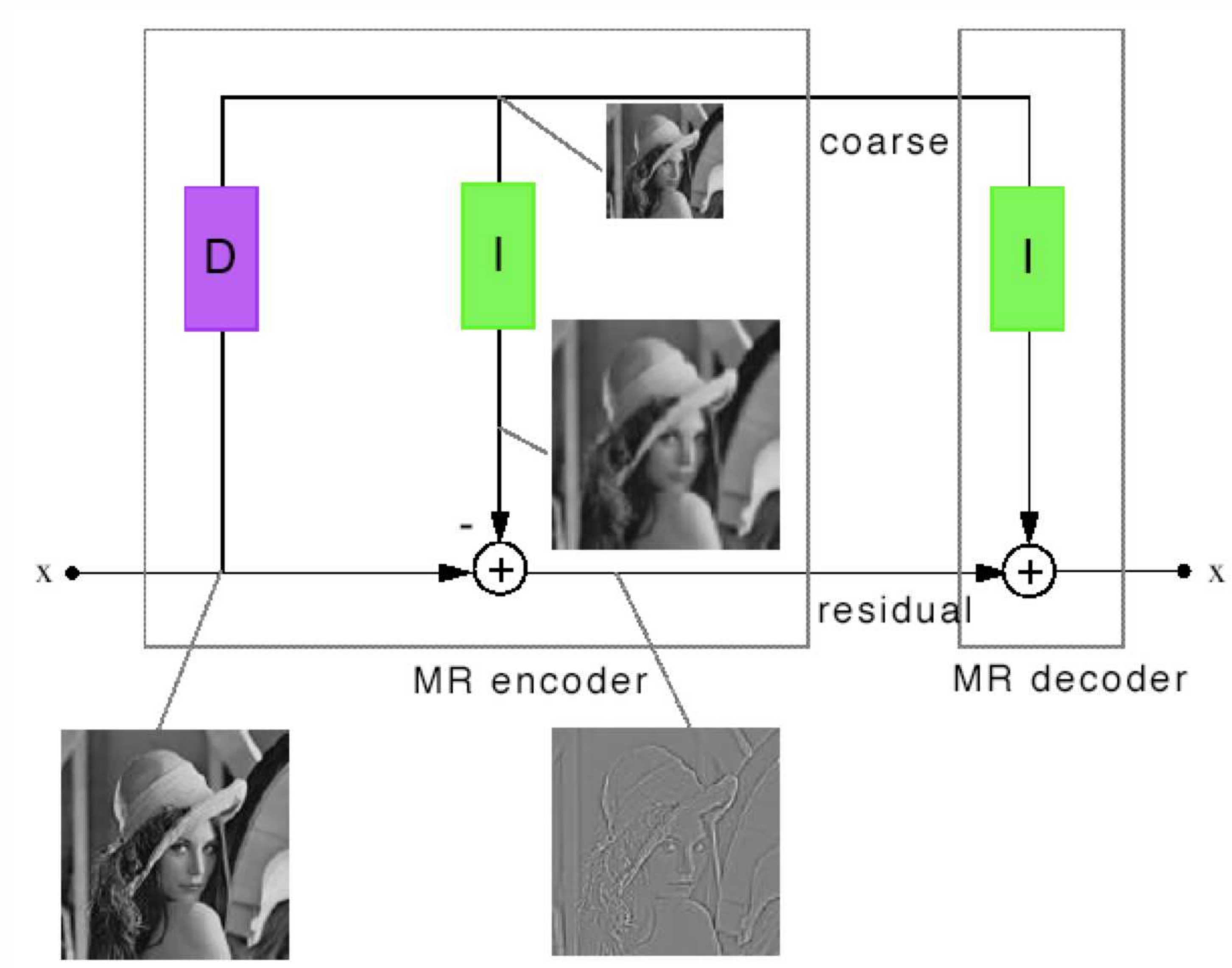


- Sampled data: $s[k] = \langle x, \phi_k \rangle$, where ϕ_k is the point spreading function (PSF) of the sensoring device at location t_k.

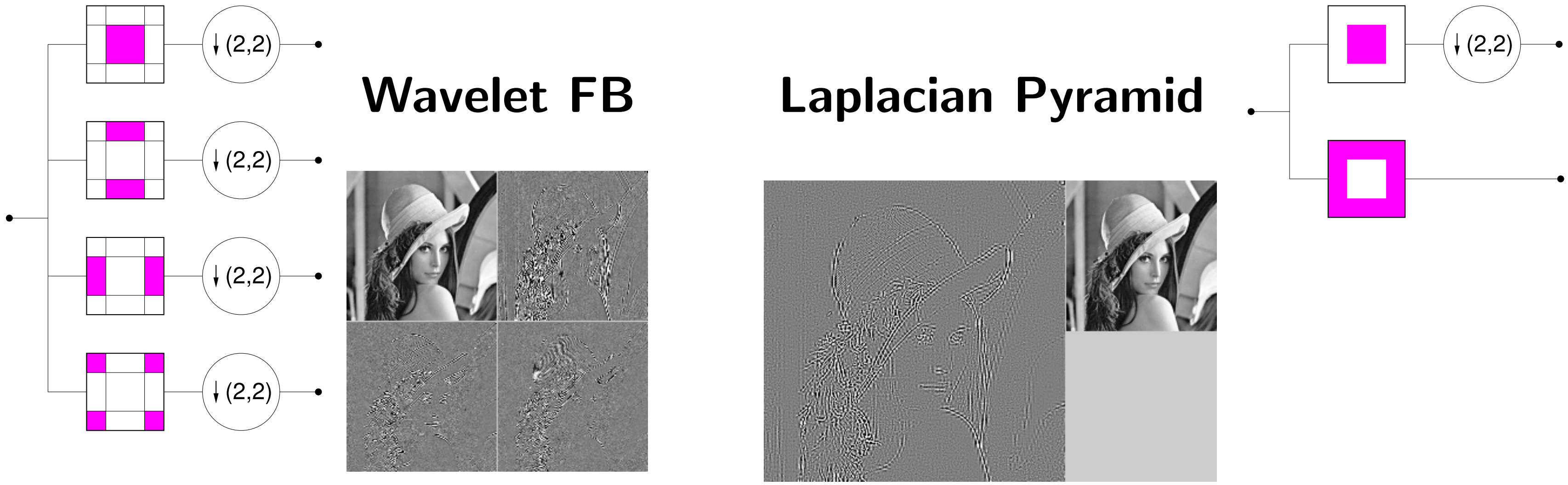
- **"Sampling theorem":** Function $x(t) \in H$ can be recovered in a numerically stable way from samples $s[k]$ if and only if $\{\phi_k\}_{k \in \Gamma}$ is a frame of H.

- Classical sampling: $H = BL([-\pi, \pi])$ and $\{\phi_k\} = \{\text{sinc}(t-k)\}_{k \in \mathbb{Z}}$

Laplacian Pyramid: Burt Adelson, 1983

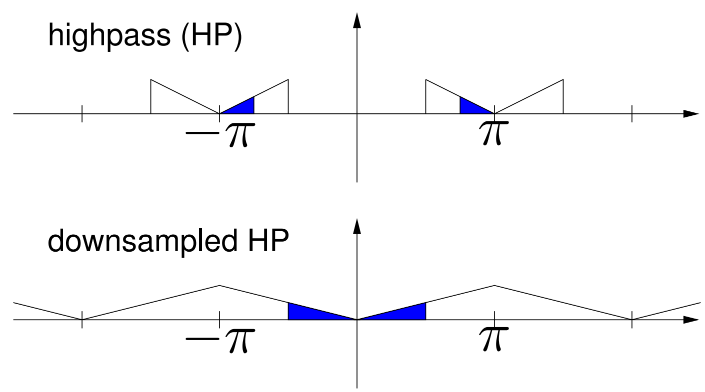


Why Laplacian Pyramid Instead of Orthogonal Filter Banks?

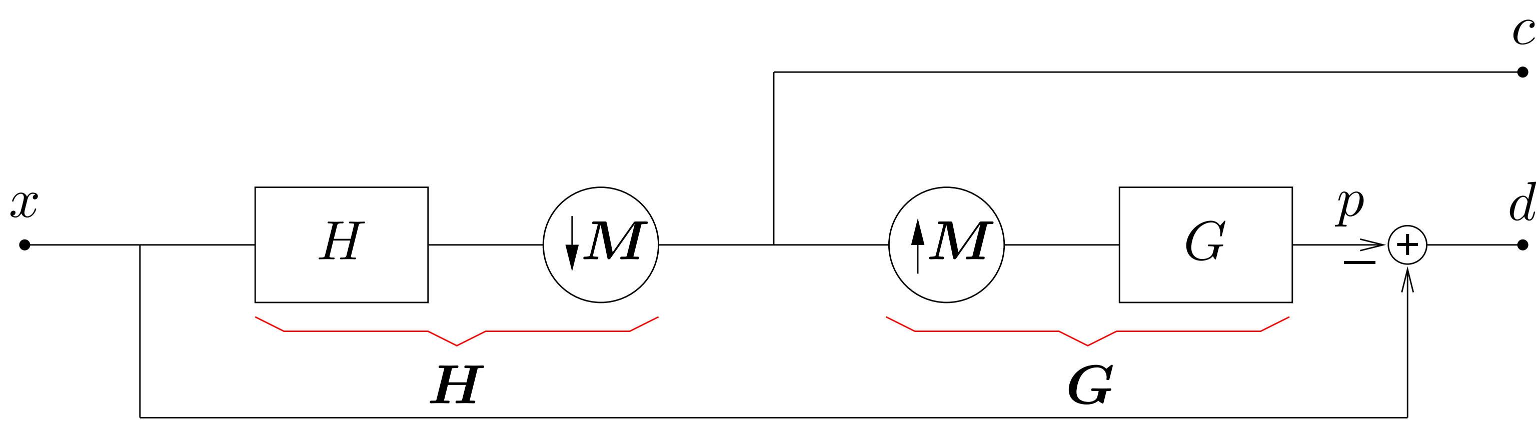


Even in higher dimensions, the Laplacian pyramid (LP) only generates one isometric detailed signal at each level.

LP has no "frequency scrambling" due to downsampling of the highpass channel:



Decomposition in the Laplacian Pyramid



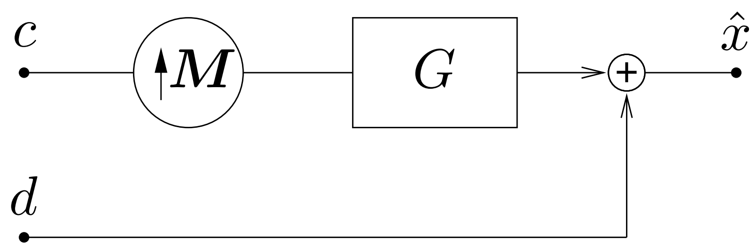
Coarse: $\boldsymbol{c} = \boldsymbol{H}\boldsymbol{x}$

Residual: $\boldsymbol{d} = \boldsymbol{x} - \boldsymbol{G}\boldsymbol{H}\boldsymbol{x} = (\boldsymbol{I} - \boldsymbol{G}\boldsymbol{H})\boldsymbol{x}.$

Combining gives

$$\underbrace{\begin{pmatrix} \boldsymbol{c} \\ \boldsymbol{d} \end{pmatrix}}_{\boldsymbol{y}} = \underbrace{\begin{pmatrix} \boldsymbol{H} \\ \boldsymbol{I} - \boldsymbol{G}\boldsymbol{H} \end{pmatrix}}_{\boldsymbol{A}} \boldsymbol{x}.$$

Usual Reconstruction in the Laplacian Pyramid



$$\hat{\boldsymbol{x}} = \underbrace{\begin{pmatrix} \boldsymbol{G} & \boldsymbol{I} \end{pmatrix}}_{\boldsymbol{S}_1} \underbrace{\begin{pmatrix} \boldsymbol{c} \\ \boldsymbol{d} \end{pmatrix}}_{\boldsymbol{y}}.$$

Note that $\boldsymbol{S}_1 \boldsymbol{A} = \boldsymbol{I}$ (perfect reconstruction) for *any* $\boldsymbol{H}$ and $\boldsymbol{G}$.

But... what about noisy pyramids: $\hat{\boldsymbol{y}} = \boldsymbol{y} + \boldsymbol{e}$?

The most serious disadvantage of the LP for coding applications [Simoncelli & Adelson, 1991]: *"...the errors from highpass subbands of a multilevel LP do not remain in these subbands but appear as broadband noise in the reconstructed signal..."*

Frame Analysis

- LP is a frame operator ($\boldsymbol{A}$) with redundancy.
- It admits an infinite number of left inverses.
- Let $\boldsymbol{S}$ be an arbitrary left inverse of $\boldsymbol{A}$,

$$\hat{\boldsymbol{x}} = \boldsymbol{S}\hat{\boldsymbol{y}} = \boldsymbol{S}(\boldsymbol{y} + \boldsymbol{e}) = \boldsymbol{x} + \boldsymbol{S}\boldsymbol{e}.$$

- The optimal left inverse (minimizing $\|\boldsymbol{S}\|$) is the pseudo-inverse of $\boldsymbol{A}$:

$$\boldsymbol{A}^{\dagger} = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T.$$

- If the noise is white, then among all left inverses, the pseudo-inverse minimizes the reconstruction MSE.
- **But**... reconstruction using the pseudo-inverse might be computationally expensive, unless we have a tight frame.

A Tight Frame Case

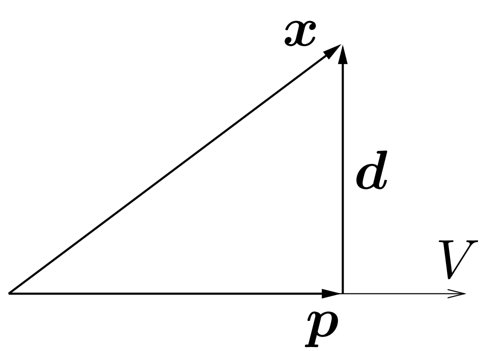
Orthogonal filters:

$$\langle g[\bullet], g[\bullet - \boldsymbol{M}n] \rangle = \delta[n], \quad \text{and}$$
$$h[n] = g[-n], \quad \text{or} \quad \boldsymbol{H} = \boldsymbol{G}^T.$$

Theorem. *The Laplacian pyramid with orthogonal filters is a tight frame.*

Proof: Under the orthogonality condition:

$$p[n] = \sum_{k \in \mathbb{Z}^d} \underbrace{\langle x[\bullet], g[\bullet - \boldsymbol{M}k] \rangle}_{c[k]} g[n - \boldsymbol{M}k].$$



Using the *Pythagorean* theorem:

$$\|\boldsymbol{x}\|^2 = \|\boldsymbol{p}\|^2 + \|\boldsymbol{d}\|^2 = \|\boldsymbol{c}\|^2 + \|\boldsymbol{d}\|^2.$$

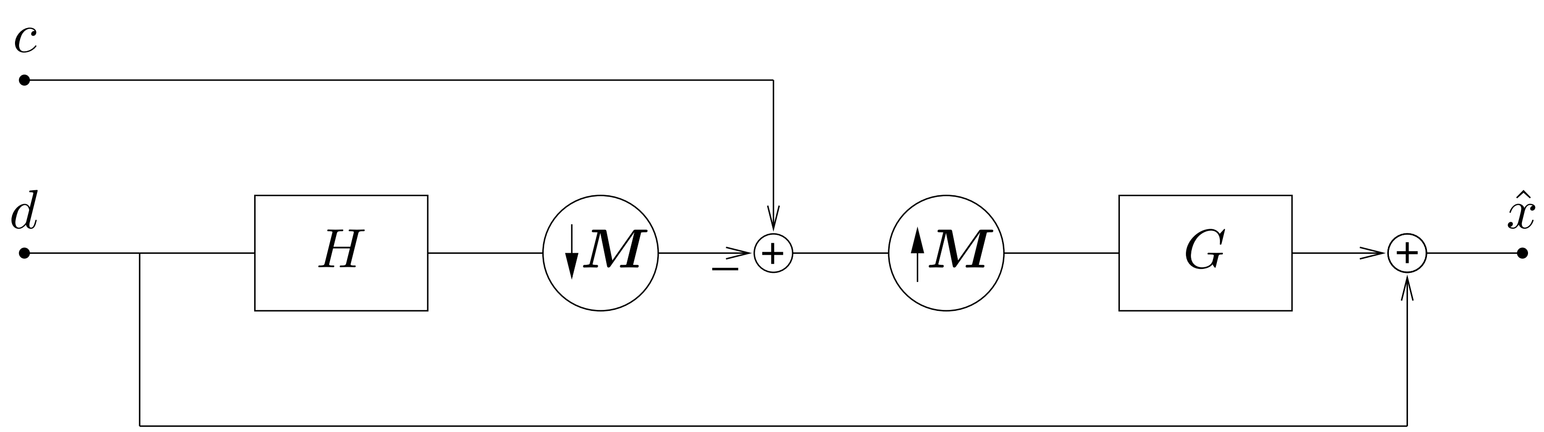
Inspiring New Reconstruction Filter Bank

As a result, pseudo-inverse of $\boldsymbol{A}$ is simply its transpose

$$\boldsymbol{A}^{\dagger} = \boldsymbol{A}^{T} = \begin{pmatrix} \boldsymbol{H} \\ \boldsymbol{I} - \boldsymbol{G}\boldsymbol{G}^{T} \end{pmatrix}^{T} = \begin{pmatrix} \boldsymbol{G} & \boldsymbol{I} - \boldsymbol{G}\boldsymbol{G}^{T} \end{pmatrix}.$$

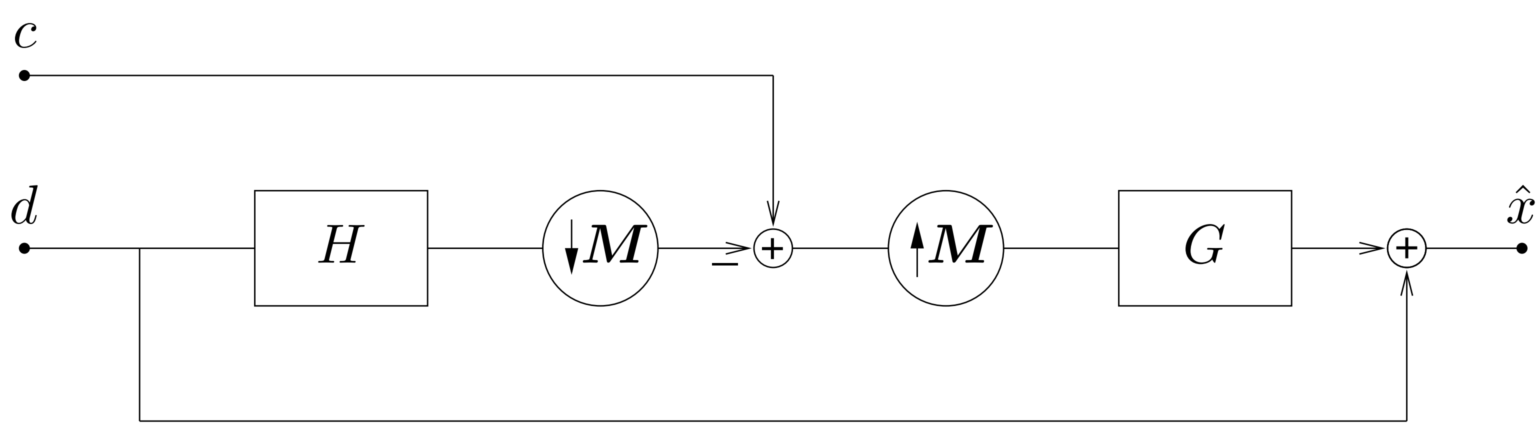
So the optimal reconstruction is

$$\hat{\boldsymbol{x}} = \boldsymbol{A}^{\dagger}\boldsymbol{y} = \boldsymbol{G}\boldsymbol{c} + (\boldsymbol{I} - \boldsymbol{G}\boldsymbol{G}^{T})\boldsymbol{d} = \boldsymbol{G}(\boldsymbol{c} - \boldsymbol{H}\boldsymbol{d}) + \boldsymbol{d}.$$



General Cases

Consider the following filter bank for reconstruction



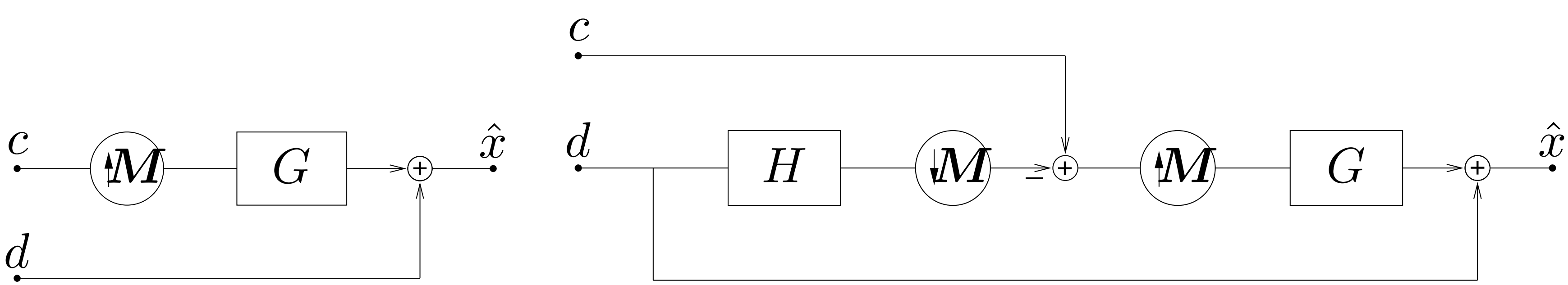
Theorem.

1. It is an inverse of the LP *if and only if* H and G are biorthogonal filters, or $\boldsymbol{GH}$ is a projector.

2. It is the pseudo-inverse *if and only if* $\boldsymbol{GH}$ is an orthogonal projector.

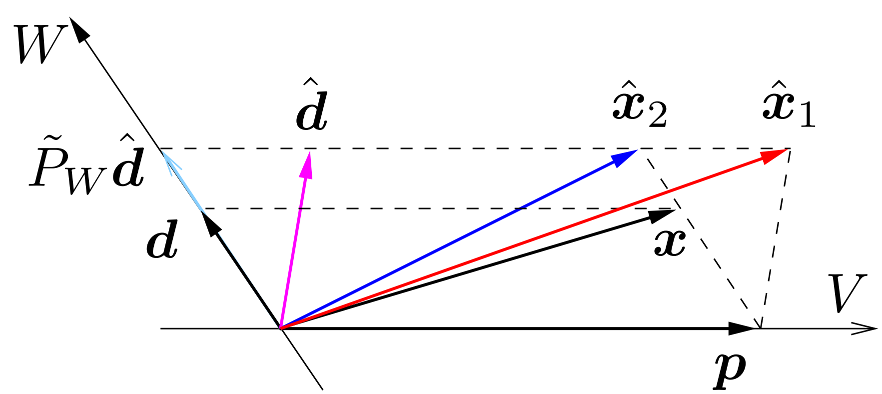
Recall: A linear operator P is a projector if $P^2 = P$.
Furthermore, if $P = P^T$ then P is an orthogonal projector.

Comparing Two Reconstruction Methods

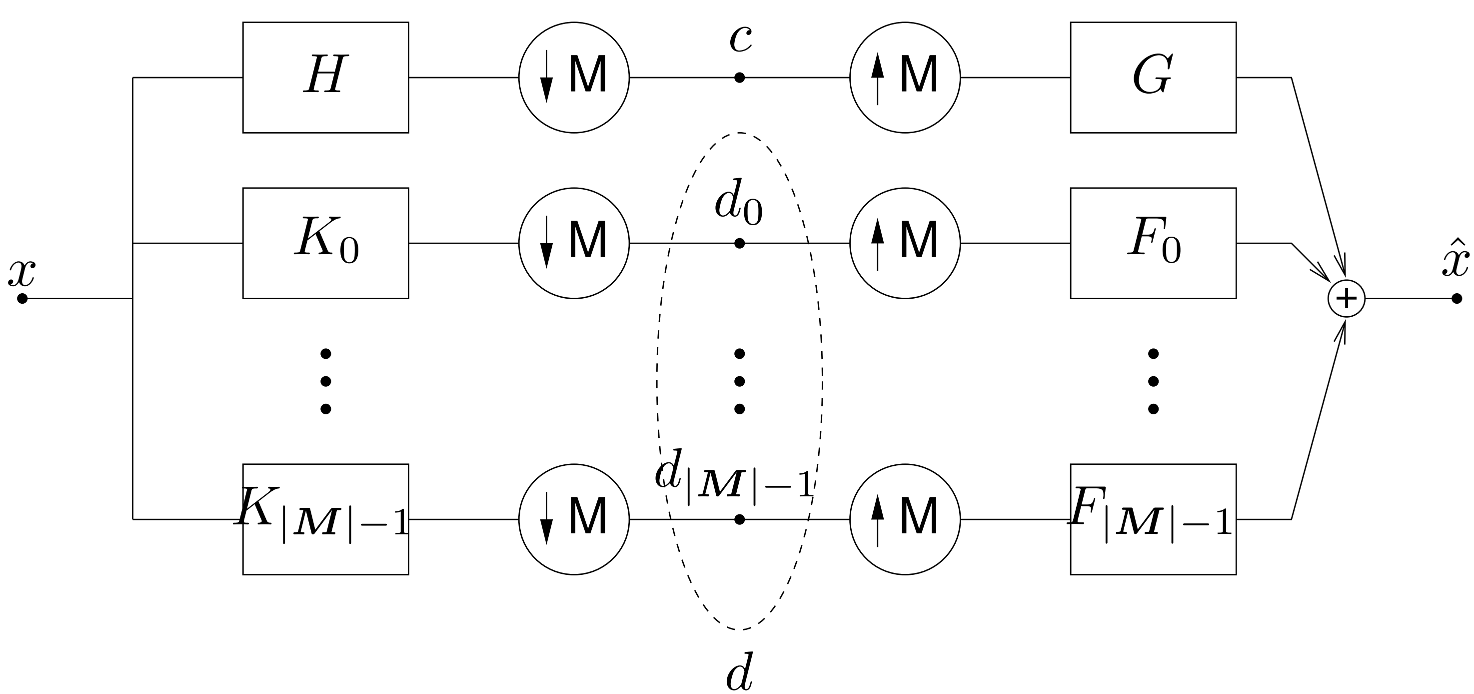


Usual reconstruction: $\boldsymbol{x}_1 = \boldsymbol{G}\boldsymbol{c} + \boldsymbol{d}$

New reconstruction: $\boldsymbol{x}_2 = \boldsymbol{G}\boldsymbol{c} + \underbrace{(\boldsymbol{I} - \boldsymbol{G}\boldsymbol{H})}_{\tilde{P}_W}\boldsymbol{d}$



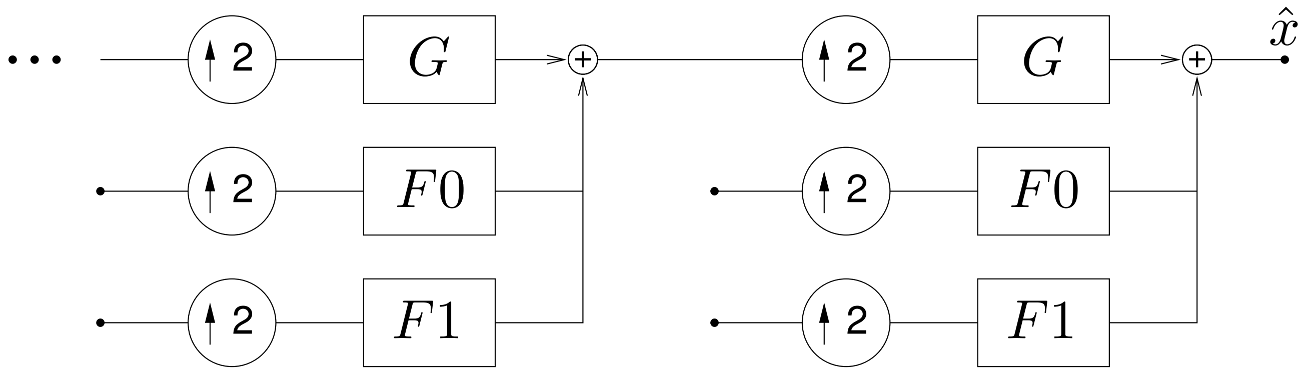
Laplacian Pyramid as an Oversampled Filter Bank



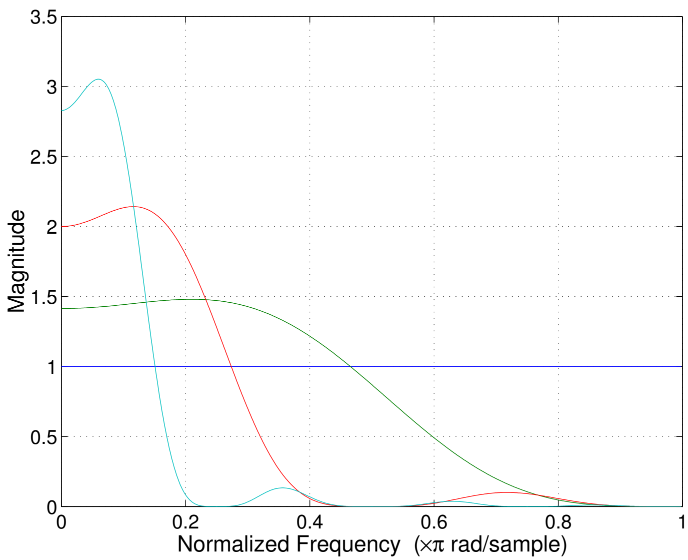
For the usual reconstruction method, synthesis filters $F_i^{[1]}$ are all-pass (delay) filters: $F_i^{[1]}(z) = z^{-k_i}$

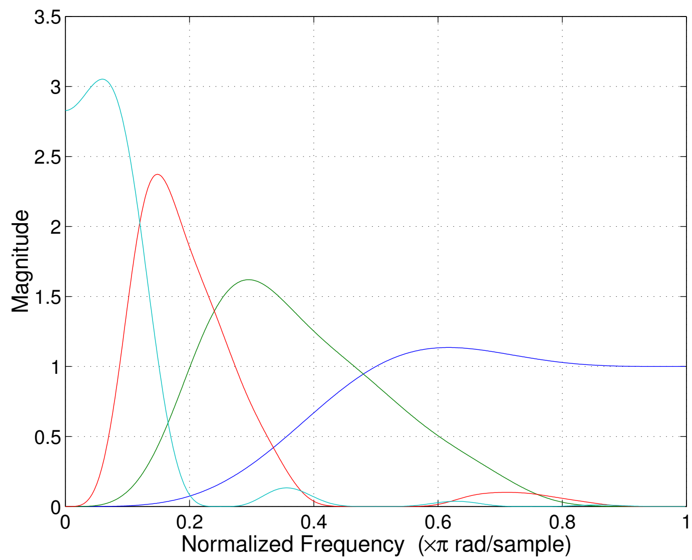
For the proposed reconstruction method, synthesis filters $F_i^{[1]}$ are high-pass filters: $F_i^{[2]}(z) = z^{-k_i} - G(z)H_i(z^{\boldsymbol{M}})$.

Multilevel Laplacian Pyramids

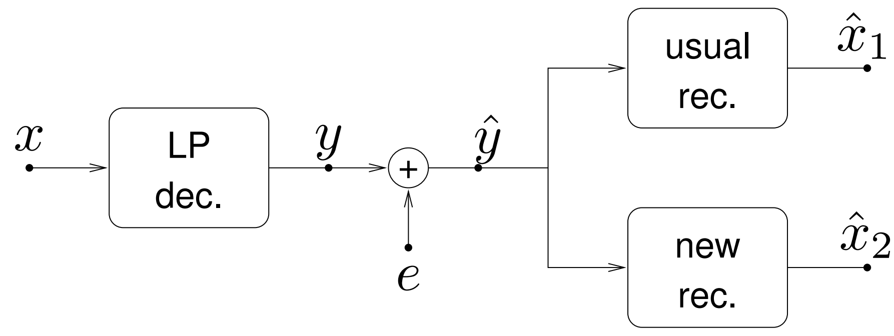


Comparing frequency responses of equivalent synthesis filters (REC-1 vs. REC-2)



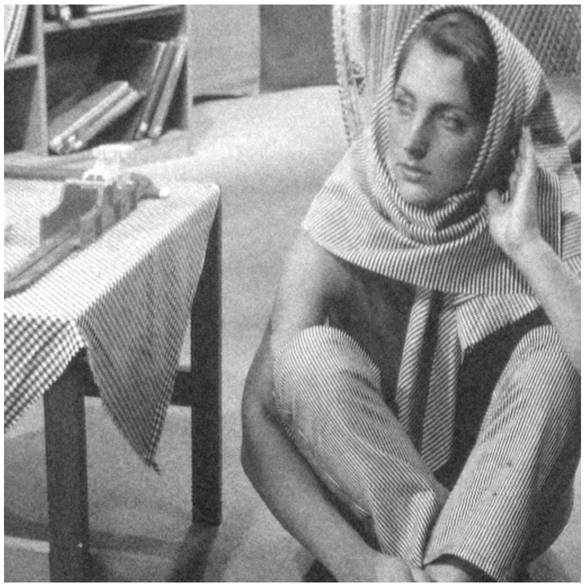


Experimental Results



With additive uniform white noise in $[0,\ 0.1]$ (*non-zero mean*)...

usual rec. SNR $=$ 6.28 dB



new rec. SNR $=$ 17.42 dB

Summary

- Frames are a powerful tool...
  - Generalizes matrix inversions for general (possible infinite dimensional) vector spaces.
  - Generalizes bases for overcomplete (redundant) systems.
- Framing pyramids lead to...
  - New reconstruction algorithm with significant improvement over the usual method.
  - Complete characterization of left inverses and the pseudo-inverse.
- Frames are everywhere...
  - Give me a linear operator with a bounded inverse, I'll frame it!
  - If you have to deal with an overcomplete system, consider the frame theory!