Special Paraunitary Matrices, Cayley Transform, and Multidimensional Orthogonal Filter Banks

Jianping Zhou, Minh N. Do, *Member, IEEE,* and Jelena Kovačević, *Fellow, IEEE*

***Abstract*— We characterize and design multidimensional orthogonal filter banks using special paraunitary matrices and the Cayley transform. Orthogonal filter banks are represented by paraunitary matrices in the polyphase domain. We define special paraunitary matrices as paraunitary matrices with unit determinant. We show that every paraunitary matrix can be characterized by a special paraunitary matrix and a phase factor. Therefore the design of paraunitary matrices (and thus of orthogonal filter banks) becomes the design of special paraunitary matrices, which requires a smaller set of nonlinear equations. Moreover, we provide a complete characterization of special paraunitary matrices in the Cayley domain, which converts nonlinear constraints into linear constraints. Our method greatly simplifies the design of multidimensional orthogonal filter banks and leads to complete characterizations of such filter banks.**

***Index Terms*— Cayley Transform, Filter Banks, Multidimensional Filter Banks, Nonseparable Filter Design, Orthogonal Filter Banks, Paraunitary, Polyphase, Special Paraunitary.**

Jianping Zhou is with the Department of Electrical and Computer Engineering and the Coordinated Science Laboratory, University of Illinois at Urbana-Champaign (email: jzhou2@uiuc.edu).

Minh N. Do is with the Department of Electrical and Computer Engineering, the Coordinated Science Laboratory, and the Beckman Institute, University of Illinois at Urbana-Champaign (email: minhdo@uiuc.edu).

Jelena Kovačević is with the Department of Biomedical Engineering and the Department of Electrical and Computer Engineering, Carnegie Mellon University (email: jelenak@cmu.edu).

This work was supported in part by the National Science Foundation under Grant CCR-0237633 (CAREER).

I. INTRODUCTION

Multidimensional (MD) filter banks have gained particular attention in the last decade [1]–[9]. Nonseparable filter banks can capture geometric structures in MD data and offer more freedom and better frequency selectivity than traditional separable filter banks constructed from one-dimensional (1D) filter banks. Nonseparable filter banks also provide flexible directional decomposition of multidimensional data [10]. Therefore, nonseparable filter banks are more suited to image and video applications.

Orthogonal filter banks are special critically sampled perfect reconstruction filter banks where the synthesis filters are time-reversals of the analysis filters. Orthogonal filter banks can be used to construct orthonormal wavelet bases [11], [12]. Because of orthogonality, orthogonal filter banks offer certain conveniences; for example, the best M-term approximation is simply done by keeping those M coefficients with largest magnitude.

Designing nonseparable MD orthogonal filter banks is a challenging task. Traditional design methods for 1D orthogonal filter banks cannot be extended to higher dimensions directly due to the lack of an MD factorization theorem. In the infinite impulse response (IIR) case, Fettweis et al. applied wave digital filters and designed a class of orthogonal filter banks [13]. In the finite impulse response (FIR) case, there are only a few design examples (for example, [3]).

In the polyphase domain, the polyphase synthesis matrix of an orthogonal filter bank is a *paraunitary* matrix, $\mathbf{U}(\boldsymbol{z})$, that satisfies

$$\mathbf{U}^T(\boldsymbol{z}^{-1})\mathbf{U}(\boldsymbol{z}) = \mathbf{I}, \quad \text{for real coefficients.} \tag{1}$$

A paraunitary matrix is an extension of a unitary matrix when the matrix entries are Laurent polynomials. Paraunitary matrices are unitary on the unit circle. For simplicity, we consider only filter banks with real coefficients. The paraunitary condition (1) requires solving a set of nonlinear equations — a difficult problem. Vaidyanathan and Hoang provided a complete characterization of paraunitary FIR matrices for 1D orthogonal filter banks via a lattice factorization [8](pp. 302–322). However, in multiple dimensions the lattice structure is *not* a complete characterization.

Recently we proposed a complete characterization of MD orthogonal filter banks using the Cayley transform and designed some orthogonal filter banks for both IIR and FIR cases [14]. The Cayley transform maps a paraunitary matrix to a *para-skew-Hermitian* matrix $\mathbf{H}(\boldsymbol{z})$ that satisfies

$$\mathbf{H}(\boldsymbol{z}^{-1}) = -\mathbf{H}^T(\boldsymbol{z}), \quad \text{for real coefficients.} \qquad (2)$$

Conversely, the inverse Cayley transform maps a para-skew-Hermitian matrix to a paraunitary matrix. Therefore, the Cayley transform establishes a one-to-one mapping between paraunitary matrices and para-skew-Hermitian matrices. A para-skew-Hermitian matrix is an extension of a skew-Hermitian matrix when the matrix entries are Laurent polynomials. Para-skew-Hermitian matrices are skew-Hermitian on the unit circle. In contrast to solving for the nonlinear paraunitary condition in (1), the para-skew-Hermitian condition amounts to *linear* constraints on the matrix entries in (2), leading to an easier design problem.

The new contribution of this paper is the introduction of the *special paraunitary* (SPU) matrix that leads to a simplified and complete characterization of MD orthogonal filter banks. A paraunitary matrix $\mathbf{U}(\boldsymbol{z})$ is said to be *special* paraunitary if its determinant equals 1. We will show that any paraunitary matrix can be characterized by an SPU matrix and a phase factor that applies to one column, as illustrated in Fig. 1. This leads to an important signal processing result that any N-channel orthogonal filter bank is completely determined by its $N-1$ synthesis filters and a phase factor in the last synthesis filter. Although this result was shown for 1D two-channel orthogonal filter banks [15] and MD two-channel orthogonal filter banks [3], to the best of our knowledge, this is the first time it is proved for general orthogonal filter banks of any dimension and any number of channels. Moreover, the design problem of orthogonal filter banks can be converted into that of SPU matrices, leading to solving a smaller set of nonlinear equations. In other words, the SPU condition provides the core of the orthogonal condition for a filter bank. Finally, since the characterization of SPU matrices in the Cayley domain is also simpler than that of the general paraunitary matrices, SPU matrices also simplify the characterization of MD orthogonal filter banks in the Cayley domain.

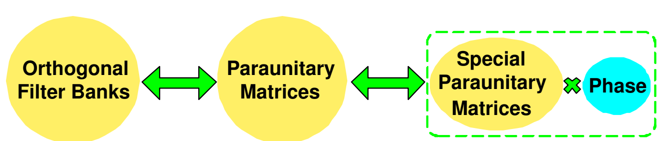


Fig. 1. Relationship among orthogonal filter banks, paraunitary matrices, and special paraunitary matrices: Orthogonal filter banks are characterized by paraunitary matrices in the polyphase domain; Paraunitary matrices are characterized by special paraunitary matrices and phase factors.

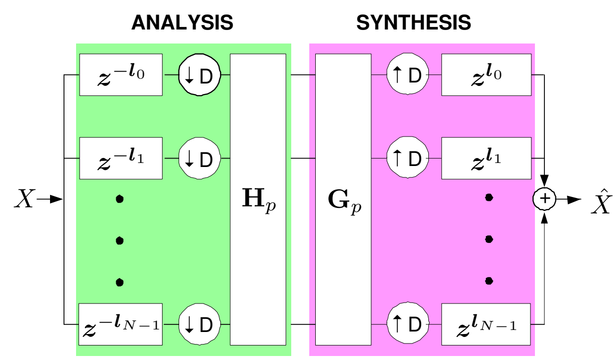
The rest of the paper is organized as follows. In Section II, we study the link between multidimensional orthogonal filter banks and special paraunitary matrices. In Section III, we study the Cayley transform of special paraunitary matrices. The characterization of two-channel special paraunitary matrices in the Cayley domain and the design of two-channel orthogonal filter banks are given in Section IV. We conclude in Section V. The glossary of abbreviations is given as follows.

CT Cayley Transform.
FCT FIR-Cayley Transform.
FIR Finite Impulse Response.
IIR Infinite Impulse Response.
MD Multi-Dimensional.
PSH Para-Skew-Hermitian.
SPSH Special Para-Skew-Hermitian.
SPU Special Paraunitary.

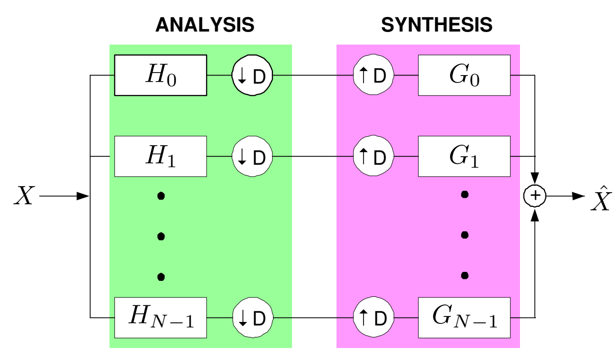
II. Multidimensional Orthogonal Filter Banks and Special Paraunitary Matrices

A. Multidimensional Orthogonal Filter Banks

We start with notations. Throughout the paper, we will always refer to M as the number of dimensions, and N as the number of channels. In MD, $\boldsymbol{z}$ stands for an M-dimensional variable $\boldsymbol{z} = [z_1, z_2, \ldots, z_M]^T$ and $\boldsymbol{z}^{-1}$ stands for $[z_1^{-1}, z_2^{-1}, \ldots, z_M^{-1}]^T$. Raising $\boldsymbol{z}$ to an M-dimensional integer vector power $\boldsymbol{k} = [k_1, k_2, \ldots, k_M]^T$ yields $\boldsymbol{z}^{\boldsymbol{k}} = \prod_{i=1}^M z_i^{k_i}$. For a matrix $\mathbf{A}$, we use $A_{i,j}$ for its entry at (i,j). We use $\mathbf{I}_N$ to denote the $N \times N$ identity matrix, and omit the subscript when it is clear from the context. For a matrix $\mathbf{A}$, the entry of its adjugate (denoted by adj$\mathbf{A}$) at (i,j) is defined as $(-1)^{i+j} \det \mathbf{A}_{j,i}$, where $\mathbf{A}_{j,i}$ is the submatrix of $\mathbf{A}$ obtained by deleting its jth row and ith column.



(a)



(b)

Fig. 2. Multidimensional filter banks and polyphase representation. (a) A multidimensional N-channel filter bank: H_i and G_i are analysis and synthesis filters, respectively; $\mathbf{D}$ is an $M \times M$ sampling matrix. (b) Polyphase representation: $\mathbf{H}_p$ and $\mathbf{G}_p$ are $N \times N$ analysis and synthesis polyphase matrices, respectively; $\{\boldsymbol{l}_j\}_{j=0}^{N-1}$ is the set of integer vectors of the form $\mathbf{D}\boldsymbol{t}$, such that $\boldsymbol{t} \in [0,1)^M$.

Consider an MD N-channel filter bank as shown in Fig. 2(a). For implementation purposes, we only consider filter banks with rational filters. We are interested in the critically sampled filter bank in which the sampling rate is equal to the number of channels, that is, $|\det \mathbf{D}| = N$. In the polyphase domain, the analysis and synthesis parts can be represented by $N \times N$ polyphase matrices $\mathbf{H}_p(\boldsymbol{z})$ and $\mathbf{G}_p(\boldsymbol{z})$ respectively, as shown in Fig. 2(b). In particular, IIR filter banks lead to IIR polyphase matrices, entries of which are rational functions, while FIR filter banks lead to FIR polyphase matrices, entries of which are polynomials. The analysis and synthesis filters are related to the corresponding polyphase matrices as:

$$H_i(\boldsymbol{z}) = \sum_{\boldsymbol{l}_j \in \mathcal{N}(\mathbf{D})} \boldsymbol{z}^{-\boldsymbol{l}_j} \{H_p\}_{i,j}(\boldsymbol{z}^{\mathbf{D}}),$$
$$G_i(\boldsymbol{z}) = \sum_{\boldsymbol{l}_j \in \mathcal{N}(\mathbf{D})} \boldsymbol{z}^{\boldsymbol{l}_j} \{G_p\}_{j,i}(\boldsymbol{z}^{\mathbf{D}}), \text{ for } i = 0, 1, \ldots, N-1, \tag{3}$$

where $\boldsymbol{z}^{\mathbf{D}} = [\boldsymbol{z}^{\boldsymbol{d}_1}, \boldsymbol{z}^{\boldsymbol{d}_2}, \ldots, \boldsymbol{z}^{\boldsymbol{d}_M}]^T$ with $\boldsymbol{d}_i$ is the ith column of $\mathbf{D}$, and $\mathcal{N}(\mathbf{D})$ is the set of integer vectors of the form $\mathbf{D}\boldsymbol{t}$, such that $\boldsymbol{t} \in [0, 1)^M$ [8](pp. 561–566).

In the polyphase domain, the perfect reconstruction condition $\hat{X}(\boldsymbol{z}) = X(\boldsymbol{z})$ is equivalent to $\mathbf{H}_p(\boldsymbol{z})\mathbf{G}_p(\boldsymbol{z}) = \mathbf{I}$. Orthogonal filter banks additionally require $\mathbf{H}_p(\boldsymbol{z}) = \mathbf{G}_p^T(\boldsymbol{z}^{-1})$, and thus $\mathbf{H}_p(\boldsymbol{z})$ and $\mathbf{G}_p(\boldsymbol{z})$ are paraunitary matrices. Therefore, designing an orthogonal filter bank boils down to designing a paraunitary matrix. From now on, we denote $\mathbf{G}_p(\boldsymbol{z})$ by $\mathbf{U}(\boldsymbol{z})$ for convenience.

B. Special Paraunitary Matrices

We define *special paraunitary* (SPU) matrices as paraunitary matrices with unit determinant. The concept of the special paraunitary matrix is similar to that of the special orthogonal matrix. An orthogonal matrix is said to be special orthogonal if its determinant equals 1. SPU matrices satisfy all the properties of paraunitary matrices. In addition, the product of two SPU matrices is also SPU. Because of the additional condition on the determinant, it would now seem as if we had one more equation to solve for the SPU matrix than for the paraunitary matrix. On the contrary, we will show that the "normalized" determinant allows us to reduce the number of nonlinear equations and thus simplify the design problem.

Theorem 1: Suppose $\mathbf{U}(\boldsymbol{z})$ is an $N \times N$ matrix and $\mathbf{U}_{N-1}(\boldsymbol{z})$ is its submatrix obtained by deleting its last column. Then $\mathbf{U}(\boldsymbol{z})$ is special paraunitary *if and only if*

$$\mathbf{U}_{N-1}^T(\boldsymbol{z}^{-1})\mathbf{U}_{N-1}(\boldsymbol{z}) = \mathbf{I}_{N-1}, \tag{4}$$

and

$$U_{i,N}(\boldsymbol{z}) = (-1)^{i+N} \det \mathbf{U}_{N-1,i}\,(\boldsymbol{z}^{-1}), \tag{5}$$

where $\mathbf{U}_{N-1,i}\,(\boldsymbol{z}^{-1})$ is the submatrix of $\mathbf{U}_{N-1}(\boldsymbol{z}^{-1})$ obtained by deleting its ith row.

Proof: See Appendix A. ■

By Theorem 1, to design an SPU matrix $\mathbf{U}(\boldsymbol{z})$, we first choose its first $N-1$ columns satisfying (4), independent of the last column. After that, we can simply compute the last column of $\mathbf{U}(\boldsymbol{z})$ from its first $N-1$ columns using (5). In other words, to solve the SPU condition, we only need solve the condition (4) instead of the paraunitary condition (1). A direct expansion of (1) generates N^2 equations. Among them, there are $N(N-1)/2$ equivalent pairs. Therefore, the paraunitary condition (1) leads to $N(N+1)/2$ equations with N^2 unknowns. The condition (4) leads to $N(N-1)/2$ equations with $N^2 - N$ unknowns. Moreover, it can be seen that the set of nonlinear equations generated by (4) is a subset of that generated by (1). Therefore, solving the SPU condition instead of the paraunitary condition saves us N nonlinear equations and N unknowns, leading to a simpler design problem. To illustrate this simplification, we consider two-channel and three-channel cases.

Example 1: Let $\mathbf{U}(\boldsymbol{z})$ be a 2×2 matrix with

$$\mathbf{U}(\boldsymbol{z}) = \begin{pmatrix} U_{00}(\boldsymbol{z}) & U_{01}(\boldsymbol{z}) \\ U_{10}(\boldsymbol{z}) & U_{11}(\boldsymbol{z}) \end{pmatrix}.$$

Then the paraunitary condition $\mathbf{U}^T(\boldsymbol{z}^{-1})\mathbf{U}(\boldsymbol{z}) = \mathbf{I}$ is equivalent to

$$\begin{cases} U_{00}(\boldsymbol{z})U_{00}(\boldsymbol{z}^{-1}) + U_{10}(\boldsymbol{z})U_{10}(\boldsymbol{z}^{-1}) = 1, \\ U_{00}(\boldsymbol{z})U_{01}(\boldsymbol{z}^{-1}) + U_{10}(\boldsymbol{z})U_{11}(\boldsymbol{z}^{-1}) = 0, \\ U_{01}(\boldsymbol{z})U_{01}(\boldsymbol{z}^{-1}) + U_{11}(\boldsymbol{z})U_{11}(\boldsymbol{z}^{-1}) = 1. \end{cases} \tag{6}$$

Solving the system involves solving 3 nonlinear equations with 4 unknowns U_{00}, U_{01}, U_{10}, U_{11}.

In contrast, if $\mathbf{U}(\boldsymbol{z})$ is SPU, then (4) becomes

$$U_{00}(\boldsymbol{z})U_{00}(\boldsymbol{z}^{-1}) + U_{10}(\boldsymbol{z})U_{10}(\boldsymbol{z}^{-1}) = 1. \tag{7}$$

After solving (7), by Theorem 1 the second column of $\mathbf{U}(\boldsymbol{z})$ can be computed as

$$U_{01}(\boldsymbol{z}) = -U_{10}(\boldsymbol{z}^{-1}) \quad \text{and} \quad U_{11}(\boldsymbol{z}) = U_{00}(\boldsymbol{z}^{-1}).$$

In other words, the complete characterization of a 2×2 SPU matrix $\mathbf{U}_s(\boldsymbol{z})$ is

$$\mathbf{U}_s(\boldsymbol{z}) = \begin{pmatrix} U_{00}(\boldsymbol{z}) & -U_{10}(\boldsymbol{z}^{-1}) \\ U_{10}(\boldsymbol{z}) & U_{00}(\boldsymbol{z}^{-1}) \end{pmatrix},$$

where $U_{00}(\boldsymbol{z})$ and $U_{10}(\boldsymbol{z})$ satisfy the power complementary property given in (7). Therefore, for 2×2 SPU matrices, we need solve only 1 nonlinear equation with 2 unknowns, instead of 3 nonlinear equations with 4 unknowns required for general 2×2 paraunitary matrices.

Example 2: Let $\mathbf{U}(\boldsymbol{z})$ be a 3×3 matrix with

$$\mathbf{U}(\boldsymbol{z}) = \begin{pmatrix} U_{00}(\boldsymbol{z}) & U_{01}(\boldsymbol{z}) & U_{02}(\boldsymbol{z}) \\ U_{10}(\boldsymbol{z}) & U_{11}(\boldsymbol{z}) & U_{12}(\boldsymbol{z}) \\ U_{20}(\boldsymbol{z}) & U_{21}(\boldsymbol{z}) & U_{22}(\boldsymbol{z}) \end{pmatrix}.$$

Then the paraunitary condition $\mathbf{U}^T(\boldsymbol{z}^{-1})\mathbf{U}(\boldsymbol{z}) = \mathbf{I}$ leads to 6 nonlinear equations with 9 unknowns.

In contrast, if $\mathbf{U}(\boldsymbol{z})$ is SPU, then (4) becomes

$$\begin{pmatrix} U_{00}(\boldsymbol{z}^{-1}) & U_{10}(\boldsymbol{z}^{-1}) & U_{20}(\boldsymbol{z}^{-1}) \\ U_{01}(\boldsymbol{z}^{-1}) & U_{11}(\boldsymbol{z}^{-1}) & U_{21}(\boldsymbol{z}^{-1}) \end{pmatrix} \begin{pmatrix} U_{00}(\boldsymbol{z}) & U_{01}(\boldsymbol{z}) \\ U_{10}(\boldsymbol{z}) & U_{11}(\boldsymbol{z}) \\ U_{20}(\boldsymbol{z}) & U_{21}(\boldsymbol{z}) \end{pmatrix}, \tag{8}$$

which amounts to 3 nonlinear equations with 6 unknowns in the first two columns of $\mathbf{U}(\boldsymbol{z})$. After solving (8), by Theorem 1, the third column of $\mathbf{U}(\boldsymbol{z})$ can be computed as

$$U_{02}(\boldsymbol{z}) = U_{10}(\boldsymbol{z}^{-1})U_{21}(\boldsymbol{z}^{-1}) - U_{11}(\boldsymbol{z}^{-1})U_{20}(\boldsymbol{z}^{-1}),$$
$$U_{12}(\boldsymbol{z}) = U_{01}(\boldsymbol{z}^{-1})U_{20}(\boldsymbol{z}^{-1}) - U_{00}(\boldsymbol{z}^{-1})U_{21}(\boldsymbol{z}^{-1}),$$
$$U_{22}(\boldsymbol{z}) = U_{00}(\boldsymbol{z}^{-1})U_{11}(\boldsymbol{z}^{-1}) - U_{01}(\boldsymbol{z}^{-1})U_{10}(\boldsymbol{z}^{-1}).$$

Therefore, for 3×3 SPU matrices, we need solve 3 nonlinear equations with 6 unknowns, instead of 6 nonlinear equations with 9 unknowns required for general 3×3 paraunitary matrices.

C. Connection between Orthogonal Filter Banks and Special Paraunitary Matrices

We just showed that designing SPU matrices is easier than designing paraunitary matrices. In this subsection, we will characterize paraunitary matrices via SPU matrices and use this characterization to simplify the design of MD orthogonal filter banks.

Proposition 1: A matrix $\mathbf{U}(\boldsymbol{z})$ is paraunitary *if and only if* it can be written as $\mathbf{U}(\boldsymbol{z}) = \mathbf{U}_s(\boldsymbol{z})\Lambda(\boldsymbol{z})$ such that $\mathbf{U}_s(\boldsymbol{z})$ is a special paraunitary matrix, and

$$\Lambda(\boldsymbol{z}) = \text{diag}\big(1, \ldots, 1, \Delta(\boldsymbol{z})\big), \tag{9}$$

where $\Delta(\boldsymbol{z}) = \det \mathbf{U}(\boldsymbol{z})$ is an allpass filter, that is, $\Delta(\boldsymbol{z})\Delta(\boldsymbol{z}^{-1}) = 1$.

Proof: Suppose that $\mathbf{U}(\boldsymbol{z})$ is a paraunitary matrix. From (1), we have

$$\det \mathbf{U}^T(\boldsymbol{z}^{-1}) \cdot \det \mathbf{U}(\boldsymbol{z}) = 1,$$

which implies that $\Delta(\boldsymbol{z}) = \det \mathbf{U}(\boldsymbol{z})$ is an allpass filter. Therefore, the diagonal matrix $\Lambda(\boldsymbol{z})$ defined as in (9) is paraunitary. Let $\mathbf{U}_s(\boldsymbol{z}) = \mathbf{U}(\boldsymbol{z})\Lambda^{-1}(\boldsymbol{z})$. Then $\mathbf{U}_s(\boldsymbol{z})$ is also paraunitary, and

$$\det \mathbf{U}_s(\boldsymbol{z}) = \det \mathbf{U}(\boldsymbol{z})\Delta^{-1}(\boldsymbol{z}) = 1,$$

which means that $\mathbf{U}_s(\boldsymbol{z})$ is SPU.

The sufficient condition is straightforward to verify. ■

For paraunitary FIR matrices, the characterization using SPU matrices can be simplified further.

Corollary 1: A matrix $\mathbf{U}(\boldsymbol{z})$ is a paraunitary FIR matrix *if and only if* it can be written as $\mathbf{U}(\boldsymbol{z}) = \mathbf{U}_s(\boldsymbol{z})\Lambda(\boldsymbol{z})$ such that $\mathbf{U}_s(\boldsymbol{z})$ is a special paraunitary FIR matrix, and

$$\Lambda(\boldsymbol{z}) = \text{diag}\big(1, \ldots, 1, c\boldsymbol{z}^{\boldsymbol{k}}\big), \tag{10}$$

where $c\boldsymbol{z}^{\boldsymbol{k}} = \det \mathbf{U}(\boldsymbol{z})$, and $c = \pm 1$ and $\boldsymbol{k}$ is an integer vector.

Proof: Suppose $\mathbf{U}(\boldsymbol{z})$ is a paraunitary FIR matrix. Then $\Delta(\boldsymbol{z}) = \det \mathbf{U}(\boldsymbol{z})$ is an FIR filter, and by Proposition 1, $\Delta(\boldsymbol{z})$ is an allpass filter. Therefore, $\Delta(\boldsymbol{z})$ must be a monomial, that is, $\Delta(\boldsymbol{z}) = c\boldsymbol{z}^{\boldsymbol{k}}$, where $c = \pm 1$ and $\boldsymbol{k}$ is an integer vector. ■

By Proposition 1, any paraunitary matrix $\mathbf{U}(\boldsymbol{z})$ can be converted into an SPU matrix, where the first $N-1$ columns of the matrix are kept the same and the last column is multiplied with the allpass filter $\big(\det \mathbf{U}(\boldsymbol{z})\big)^{-1}$. Now we can directly apply the characterization of SPU matrices in Theorem 1 to paraunitary matrices.

Theorem 2: Suppose $\mathbf{U}(\boldsymbol{z})$ is an $N \times N$ matrix and $\mathbf{U}_{N-1}(\boldsymbol{z})$ is its submatrix obtained by deleting its last column. Then $\mathbf{U}(\boldsymbol{z})$ is paraunitary *if and only if*

$$\mathbf{U}_{N-1}^T(\boldsymbol{z}^{-1})\mathbf{U}_{N-1}(\boldsymbol{z}) = \mathbf{I}_{N-1},$$

and

$$U_{i,N}(\boldsymbol{z}) = (-1)^{i+N}\Delta(\boldsymbol{z}) \det \mathbf{U}_{N-1,i}\,(\boldsymbol{z}^{-1}), \tag{11}$$

where $\Delta(\boldsymbol{z}) = \det \mathbf{U}(\boldsymbol{z})$ is an allpass filter, and $\mathbf{U}_{N-1,i}\,(\boldsymbol{z}^{-1})$ is the submatrix of $\mathbf{U}_{N-1}(\boldsymbol{z}^{-1})$ obtained by deleting its ith row.

Proof: The proof directly follows from Theorem 1 and Proposition 1. ■

By Theorem 2, an $N \times N$ paraunitary matrix is completely determined by its first $N-1$ columns and an allpass filter. This result can be seen as the extension of that of the unitary matrix: an $N \times N$ unitary matrix is completely determined by its $N-1$ columns (up to a unit-norm factor). To illustrate Theorem 2, we consider the two-channel case.

Example 3: By Theorem 2, a 2×2 paraunitary matrix $\mathbf{U}(\boldsymbol{z})$ can be written as

$$\mathbf{U}(\boldsymbol{z}) = \begin{pmatrix} U_{00}(\boldsymbol{z}) & -U_{10}(\boldsymbol{z}^{-1})\Delta(\boldsymbol{z}) \\ U_{10}(\boldsymbol{z}) & U_{00}(\boldsymbol{z}^{-1})\Delta(\boldsymbol{z}) \end{pmatrix},$$

where $U_{00}(\boldsymbol{z})$ and $U_{10}(\boldsymbol{z})$ satisfy the power complementary property in (7), and $\Delta(\boldsymbol{z}) = \det \mathbf{U}(\boldsymbol{z})$ is an allpass filter. For the 1D case, Herley and Vetterli showed a similar result in [15]. For the FIR case, $\Delta(\boldsymbol{z}) = c\boldsymbol{z}^{\boldsymbol{k}}$, and Kovačević and Vetterli showed a similar result in [3]. Theorem 2 generalizes these results to any dimensions and any number of channels.

Multidimensional orthogonal filter banks are characterized by paraunitary matrices in the polyphase domain. By (3), the last synthesis filter can be written as

$$G_N(\boldsymbol{z}) = \sum_{\boldsymbol{l}_i \in \mathcal{N}(\mathbf{D})} \boldsymbol{z}^{\boldsymbol{l}_i} U_{i,\,N}(\boldsymbol{z}^{\mathbf{D}}), \tag{12}$$

where $\mathbf{D}$ is the sampling matrix used in the orthogonal filter bank. Combining (12) with (11), we have

$$G_N(\boldsymbol{z}) = \Delta(\boldsymbol{z}^{\mathbf{D}}) \sum_{\boldsymbol{l}_i \in \mathcal{N}(\mathbf{D})} (-1)^{i+N} \boldsymbol{z}^{\boldsymbol{l}_i} \det \mathbf{U}_{N-1,i}\,(\boldsymbol{z}^{-\mathbf{D}}), \tag{13}$$

which means that $G_N(\boldsymbol{z})$ is completely determined by $\mathbf{U}_{N-1}(\boldsymbol{z})$ and $\Delta(\boldsymbol{z}^{\mathbf{D}})$. Moreover, $\Delta(\boldsymbol{z}^{\mathbf{D}})$ is also a phase factor since $\Delta(\boldsymbol{z})$ is a phase factor. Since an allpass filter has magnitude gain of unity, passing an allpass system just changes the phase [16](pp. 234–240). Therefore, connecting orthogonal filter banks with paraunitary polyphase matrices using (13), we obtain the following result.

Corollary 2: Any multidimensional N-channel orthogonal filter bank is characterized by either of the following:

1) a special paraunitary matrix and a phase factor;
2) $N-1$ synthesis filters and a phase factor.

For an orthogonal FIR filter bank, this phase factor is a pure delay.

Corollary 2 has an intuitive geometric interpretation. The N synthesis filters can be seen as N orthonormal vectors in an N-dimensional vector space. Once the first $N-1$ orthonormal vectors are given, the last orthonormal vector will be completely determined (up to a unit-norm factor).

III. Cayley Transform and Special Paraunitary Matrices

A. Cayley Transform of Paraunitary Matrices

In this subsection, we briefly review the main results on the characterization of paraunitary matrices via the Cayley transform [14] that will be used later in the paper.

The *Cayley transform* (CT) of a matrix $\mathbf{U}(\boldsymbol{z})$ is defined as

$$\mathbf{H}(\boldsymbol{z}) = \big(\mathbf{I} + \mathbf{U}(\boldsymbol{z})\big)^{-1}\big(\mathbf{I} - \mathbf{U}(\boldsymbol{z})\big). \tag{14}$$

The inverse CT is itself, that is,

$$\mathbf{U}(\boldsymbol{z}) = \big(\mathbf{I} + \mathbf{H}(\boldsymbol{z})\big)^{-1}\big(\mathbf{I} - \mathbf{H}(\boldsymbol{z})\big). \tag{15}$$

The CT maps a paraunitary matrix to a para-skew-Hermitian (PSH) matrix [14], [17]. Conversely, the CT maps a PSH matrix to a paraunitary matrix. Since the PSH condition (2) amounts to *linear* constraints on the matrix entries, while the paraunitary condition (1) amounts to *nonlinear* ones, designing PSH matrices is easier than designing paraunitary matrices.

The CT greatly simplifies the design of general orthogonal filter banks, especially IIR filter banks with rational filters. However, the CT cannot be directly used to design orthogonal FIR filter banks since the FIR property is destroyed in the transform. Assume that $\mathbf{U}(\boldsymbol{z})$ is a paraunitary FIR matrix and $\mathbf{H}(\boldsymbol{z})$ is the CT of $\mathbf{U}(\boldsymbol{z})$. In general, $\mathbf{H}(\boldsymbol{z})$ is IIR. Let $D(\boldsymbol{z})$ and $\mathbf{H}'(\boldsymbol{z})$ be

$$D(\boldsymbol{z}) \overset{\text{def}}{=} 2^{-N+1} \det\big(\mathbf{I} + \mathbf{U}(\boldsymbol{z})\big), \tag{16}$$

$$\mathbf{H}'(\boldsymbol{z}) \overset{\text{def}}{=} 2^{-N+1} \text{adj}\big(\mathbf{I} + \mathbf{U}(\boldsymbol{z})\big)\big(\mathbf{I} - \mathbf{U}(\boldsymbol{z})\big). \tag{17}$$

Then $\mathbf{H}(\boldsymbol{z})$ can be represented as the quotient of the FIR matrix $\mathbf{H}'(\boldsymbol{z})$ and the FIR filter $D(\boldsymbol{z})$.

Theorem 3: ([14]) The Cayley transform of a matrix $\mathbf{H}(\boldsymbol{z})$ is a paraunitary FIR matrix *if and only if* $\mathbf{H}(\boldsymbol{z})$ can be written as $\mathbf{H}(\boldsymbol{z}) = \mathbf{H}'(\boldsymbol{z})/D(\boldsymbol{z})$, where $D(\boldsymbol{z})$ is an FIR filter and $\mathbf{H}'(\boldsymbol{z})$ is an FIR matrix, and they satisfy the following four conditions:

1) $D(\boldsymbol{z}^{-1}) = c\boldsymbol{z}^{\boldsymbol{k}} D(\boldsymbol{z})$;
2) $\mathbf{H}'^{T}(\boldsymbol{z}^{-1}) = -c\boldsymbol{z}^{\boldsymbol{k}}\mathbf{H}'(\boldsymbol{z})$;
3) $2\,D(\boldsymbol{z})^{N-1} = \det\big(D(\boldsymbol{z})\,\mathbf{I} + \mathbf{H}'(\boldsymbol{z})\big)$;
4) $D(\boldsymbol{z})^{N-2}$ is a common factor of all minors of $D(\boldsymbol{z})\,\mathbf{I} + \mathbf{H}'(\boldsymbol{z})$.

Moreover, the Cayley transform of $\mathbf{H}(\boldsymbol{z})$ can be written as

$$\mathbf{U}(\boldsymbol{z}) = \frac{\text{adj}\big(D(\boldsymbol{z})\,\mathbf{I} + \mathbf{H}'(\boldsymbol{z})\big)}{D(\boldsymbol{z})^{N-2}} - \mathbf{I}, \tag{18}$$

and the determinant of $\mathbf{U}(\boldsymbol{z})$ equals $c\boldsymbol{z}^{\boldsymbol{k}}$. ■

The characterization of SPU FIR matrices in the Cayley domain can be obtained from Theorem 3 by noting that $c\boldsymbol{z}^{\boldsymbol{k}} = \det \mathbf{U}(\boldsymbol{z}) = 1$ in Theorem 3 when $\mathbf{U}(\boldsymbol{z})$ is SPU. In other words, for the characterization of SPU FIR matrices, Conditions (1) and (2) in Theorem 3 become

$$D(\boldsymbol{z}^{-1}) = D(\boldsymbol{z}), \tag{20}$$

$$\mathbf{H}'^{T}(\boldsymbol{z}^{-1}) = -\mathbf{H}'(\boldsymbol{z}). \tag{21}$$

Therefore, $D(\boldsymbol{z})$ is a symmetric FIR filter and $\mathbf{H}'(\boldsymbol{z})$ is a PSH FIR matrix.

B. Cayley Transform of Special Paraunitary Matrices

The Cayley transform of paraunitary matrices are PSH matrices. Now we consider the Cayley transform of SPU matrices. We define a *special PSH* (SPSH) matrix as the Cayley transform of a SPU matrix. The following proposition characterizes SPSH matrices.

Proposition 2: A PSH matrix $\mathbf{H}(\boldsymbol{z})$ is special PSH *if and only if* it satisfies

$$\det\big(\mathbf{I} + \mathbf{H}(\boldsymbol{z})\big) = \det\big(\mathbf{I} - \mathbf{H}(\boldsymbol{z})\big). \tag{19}$$

Proof: Let $\mathbf{U}(\boldsymbol{z})$ be the CT of $\mathbf{H}(\boldsymbol{z})$. Since $\mathbf{H}(\boldsymbol{z})$ is PSH, $\mathbf{U}(\boldsymbol{z})$ is paraunitary. The paraunitary matrix $\mathbf{U}(\boldsymbol{z})$ is SPU if and only if its determinant equals 1. By (15),

$$\det \mathbf{U}(\boldsymbol{z}) = \frac{\det\big(\mathbf{I} - \mathbf{H}(\boldsymbol{z})\big)}{\det\big(\mathbf{I} + \mathbf{H}(\boldsymbol{z})\big)} = 1.$$

■

IV. Two-Channel Special Paraunitary Matrices

A. Complete Characterization

Proposition 2 gives the complete characterization of SPSH matrices, which are the CT's of SPU matrices. In general, this characterization is hard to use. Among MD orthogonal filter banks, the two-channel ones are the simplest and most popular. In this case, we can greatly simplify the characterization and design of SPU matrices.

Proposition 3: A 2×2 para-skew-Hermitian matrix is special para-skew-Hermitian *if and only if* its trace equals 0.

Proof: We first point out a simple but useful result, which will be used later. For a 2×2 matrix $\mathbf{A}$ and a scalar α, it is easy to verify that

$$\det(\alpha \mathbf{I} + \mathbf{A}) = \alpha^2 + \alpha\,\text{tr}\,(\mathbf{A}) + \det \mathbf{A}. \tag{22}$$

In the two-channel case, condition (19) is equivalent to

$$1 + \text{tr}(\mathbf{H}(\boldsymbol{z})) + \det(\mathbf{H}(\boldsymbol{z})) = 1 - \text{tr}(\mathbf{H}(\boldsymbol{z})) + \det(\mathbf{H}(\boldsymbol{z})),$$

which leads to $\text{tr}\,(\mathbf{H}(\boldsymbol{z})) = 0$. ■

Let $\mathbf{H}(\boldsymbol{z})$ be a 2×2 PSH matrix with

$$\mathbf{H}(\boldsymbol{z}) = \begin{pmatrix} H_{00}(\boldsymbol{z}) & H_{01}(\boldsymbol{z}) \\ H_{10}(\boldsymbol{z}) & H_{11}(\boldsymbol{z}) \end{pmatrix}.$$

Then the PSH condition (that is, the characterization of the paraunitary matrix in the Cayley domain) $\mathbf{H}(\boldsymbol{z}^{-1}) = -\mathbf{H}^{T}(\boldsymbol{z})$ becomes

$$\begin{cases} H_{00}(\boldsymbol{z}^{-1}) = -H_{00}(\boldsymbol{z}), \\ H_{11}(\boldsymbol{z}^{-1}) = -H_{11}(\boldsymbol{z}), \\ H_{01}(\boldsymbol{z}^{-1}) = -H_{10}(\boldsymbol{z}). \end{cases} \tag{23}$$

Here, we do not need to solve any nonlinear equations in the design as in (6) or (7). Instead, we only need design two antisymmetric filters, $H_{00}(\boldsymbol{z})$ and $H_{11}(\boldsymbol{z})$, independently and then choose one arbitrary filter $H_{10}(\boldsymbol{z})$ leading to $H_{01}(\boldsymbol{z})$ as in the last equation of (23). Then the problem of designing $\mathbf{H}(\boldsymbol{z})$ converts to that of designing two antisymmetric filters and one arbitrary filter. Antisymmetric filters can be formulated in the following proposition.

Proposition 4: ([14]) Suppose $W(\boldsymbol{z})$ is a multidimensional IIR filter given by $W(\boldsymbol{z}) = A(\boldsymbol{z})/B(\boldsymbol{z})$, where $A(\boldsymbol{z})$ and $B(\boldsymbol{z})$ are coprime polynomials. Then $W(\boldsymbol{z}^{-1}) = -W(\boldsymbol{z})$ *if and only if*

$$A(\boldsymbol{z}^{-1}) = c\boldsymbol{z}^{\boldsymbol{m}} A(\boldsymbol{z}) \quad \text{and} \quad B(\boldsymbol{z}^{-1}) = -c\boldsymbol{z}^{\boldsymbol{m}} B(\boldsymbol{z}),$$

where $\boldsymbol{m}$ is an arbitrary integer vector, and $c = \pm 1$.

By Proposition 3, the characterization of the SPU matrix in the Cayley domain (that is, SPSH matrix) is even simpler:

$$\mathbf{H}(\boldsymbol{z}) = \begin{pmatrix} H_{00}(\boldsymbol{z}) & H_{01}(\boldsymbol{z}) \\ -H_{01}(\boldsymbol{z}^{-1}) & -H_{00}(\boldsymbol{z}) \end{pmatrix}, \tag{24}$$

where $H_{00}(\boldsymbol{z}^{-1}) = -H_{00}(\boldsymbol{z})$. Here we only need design one antisymmetric filter, $H_{00}(\boldsymbol{z})$, and one arbitrary filter, $H_{10}(\boldsymbol{z})$. As for PSH matrices, the set of 2×2 SPSH matrices is a *linear space* (that is, if $\mathbf{A}(\boldsymbol{z})$ and $\mathbf{B}(\boldsymbol{z})$ are 2×2 SPSH matrices, then $\alpha\mathbf{A}(\boldsymbol{z}) + \beta\,\mathbf{B}(\boldsymbol{z})$ is also a 2×2 SPSH matrix for any real numbers α and β). Therefore, the CT maps the nonlinear set of 2×2 SPU matrices to the linear space of 2×2 SPSH matrices as shown in Fig. 3.

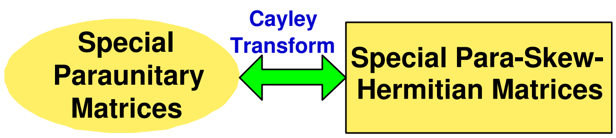


Fig. 3. One-to-one mapping between 2×2 special paraunitary matrices and 2×2 special PSH matrices. Here the rectangle stands for a linear set, while the ellipse stands for a nonlinear set.

The design process of two-channel orthogonal filter banks using the SPU and the CT is given as follows.

1) Parameterize one antisymmetric filter $H_{00}(z_1, z_2)$ by Proposition 4, and one arbitrary filter $H_{01}(z_1, z_2)$.
2) Compute the CT of the SPSH matrix generated by these two filters as given in (24) and then compute two synthesis filters from the polyphase matrix in terms of (3).
3) Impose some conditions (for example, the vanishing-moment condition in Section IV-C) on the filters and solve for the parameters.

B. FIR Characterization

As mentioned in Section III, the CT of an SPU FIR matrix is generally IIR since the CT destroys the FIR property. The CT of an SPU FIR matrix is an SPSH IIR matrix, which is the quotient of a PSH FIR matrix $\mathbf{H}'(\boldsymbol{z})$ as defined in (17) and a symmetric FIR filter $D(\boldsymbol{z})$ as defined in (16). In the two-channel case, we can simplify this characterization. In particular, since $N = 2$, Condition (4) in Theorem 3 is always satisfied. Moreover, we find a modified version of the CT that maps an SPU FIR matrix into an SPSH FIR matrix; that means it preserves the FIR property in both domains.

Definition 1: The *FIR-Cayley* transform (FCT) of a matrix $\mathbf{U}(\boldsymbol{z})$ is defined as

$$\mathbf{H}'(\boldsymbol{z}) = 2^{-1}\,\mathrm{adj}\,\big(\mathbf{I} + \mathbf{U}(\boldsymbol{z})\big)\,\big(\mathbf{I} - \mathbf{U}(\boldsymbol{z})\big). \tag{25}$$

The FCT not only preserves the FIR property, but also maps an SPU matrix into an SPSH matrix.

Proposition 5: Suppose $\mathbf{U}(\boldsymbol{z})$ is a 2×2 special paraunitary matrix and $\mathbf{H}'(\boldsymbol{z})$ is its FIR-Cayley transform. Then $\mathbf{H}'(\boldsymbol{z})$ is special para-skew-Hermitian.

Proof: By (21), $\mathbf{H}'(\boldsymbol{z})$ is PSH. To prove $\mathbf{H}'(\boldsymbol{z})$ is SPSH, by Proposition 3 we only need prove that the trace of $\mathbf{H}'(\boldsymbol{z})$ equals 0.

Let $\mathbf{H}(\boldsymbol{z})$ be the CT of $\mathbf{U}(\boldsymbol{z})$. Then $\mathbf{H}(\boldsymbol{z})$ is an SPSH matrix and its trace equals 0. By Theorem 3, $\mathbf{H}(\boldsymbol{z})$ can be written as $D(\boldsymbol{z})^{-1}\mathbf{H}'(\boldsymbol{z})$, where $D(\boldsymbol{z})$ is a filter as defined in (16). Thus, $\mathbf{H}'(\boldsymbol{z}) = D(\boldsymbol{z})\mathbf{H}(\boldsymbol{z})$, which implies that the trace of $\mathbf{H}'(\boldsymbol{z})$ also equals 0. ∎

Using the FCT, we map an SPU FIR matrix to an SPSH FIR matrix. As for SPSH IIR matrices, the set of SPSH FIR matrices is a linear space and its characterization is given in (24). The design of SPSH FIR matrices is easier than that of SPU FIR matrices. To use the FCT in the design of SPU FIR matrix, we need compute the inverse FCT.

Proposition 6: Suppose $\mathbf{H}'(\boldsymbol{z})$ is a 2×2 special para-skew-Hermitian FIR matrix. Then its FIR-Cayley transform is special paraunitary FIR *if and only if* $\big(1 - \det\mathbf{H}'(\boldsymbol{z})\big)^{1/2}$ is FIR. Moreover, the FIR-Cayley transform of $\mathbf{H}'(\boldsymbol{z})$ can be written as

$$\mathbf{U}(\boldsymbol{z}) = \pm\big(1 - \det\,\mathbf{H}'(\boldsymbol{z})\big)^{1/2}\,\mathbf{I} - \mathbf{H}'(\boldsymbol{z}).$$

Proof: See Appendix B. ∎

According to Proposition 6, the complete characterization of 2×2 SPU FIR matrices in the FCT domain is a subset of SPSH FIR matrices and they satisfy the condition given in Proposition 6. Generally, it is difficult to apply Proposition 6 in the design. To design an orthogonal FIR filter bank, we use Theorem 3 with simplifications by (20) and (21). For design details and examples, please see [14].

C. Vanishing-Moment Condition and Quincunx Orthogonal Filter Bank Design

Quincunx sampling is 2D density-2 sampling, leading to the two-channel case. Of all MD sampling patterns, the quincunx one is the most common. However, since the sampling is nonseparable, the design offers challenges. In the following, we consider the design of quincunx orthogonal IIR filter banks. It is straightforward to extend the design method to higher dimensions. For the two-dimensional quincunx sampling, its sampling matrix and the integer vectors in (3) can be written as

$$\mathbf{D} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{l}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\ \boldsymbol{l}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

In the context of wavelet design, the vanishing-moment condition plays an essential role. This condition requires the highpass filter to have Lth order zero derivatives at $\boldsymbol{z} = [1, 1]^T$. For the two-channel case, this condition is equivalent to requiring the lowpass filter to have Lth order zero derivatives at $\boldsymbol{z} = [-1, -1]^T$:

$$\left.\frac{\partial^n G_0(z_1, z_2)}{\partial z_1^i \partial z_2^{n-i}}\right|_{(-1,-1)} = 0, \tag{26}$$
$$\text{for} \quad n = 0, 1, \ldots, L-1;\ i = 0, 1, \ldots, n.$$

Moreover, for rational filters, requiring $G_0(\boldsymbol{z})$ to have Lth order zero derivatives is equivalent to requiring its numerator to have Lth order zero derivatives.

In terms of (3), the lowpass filter $G_0(\boldsymbol{z})$ becomes

$$G_0(z_1, z_2) = U_{00}(z_1 z_2, z_1 z_2^{-1}) + z_1 U_{10}(z_1 z_2, z_1 z_2^{-1}). \tag{27}$$

By (27) and (15), we can relate $G_0(\boldsymbol{z})$ (and thus its numerator) to the entries of $\mathbf{H}(\boldsymbol{z})$. By (24), the vanishing-moment condition imposes certain constraints on the derivatives of $H_{00}(\boldsymbol{z})$ and $H_{01}(\boldsymbol{z})$ at $\boldsymbol{z} = [-1, -1]^T$. If we parameterize these two filters, then the vanishing-moment condition amounts to a set of quadratic equations.

Example 4: To illustrate the design, we design an orthogonal IIR filter with second-order vanishing moment. In terms of (26), the number of vanishing moments is 3 and hence the number of free variables in the parametrization for $\mathbf{H}(\boldsymbol{z})$ is also 3. In this example, we choose $\mathbf{H}(z_1, z_2)$ to be FIR. The design procedure is as follows:

1) Parameterize $\mathbf{H}(\boldsymbol{z})$ as given in (24) with $H_{00}(z_1, z_2) = a_1(z_1 - z_1^{-1})$, and $H_{01}(z_1, z_2) = a_2 + a_3 z_2$.
2) Take the CT of $\mathbf{H}(\boldsymbol{z})$ to obtain $\mathbf{G}_p(\boldsymbol{z}) = \mathbf{U}(\boldsymbol{z})$ and impose the second-order vanishing-moment condition on the lowpass filter $G_0(\boldsymbol{z})$ as in (26), which leads to

$$\begin{cases} 2a_1 + a_2 = 0, \\ 2a_1 + a_3 = 0, \\ -1 + a_2^2 + 2a_3 + a_3^2 + 2a_2(1 + a_3) = 0. \end{cases}$$

3) Obtain the solutions:

$$a_1 = (1 \pm \sqrt{2})/4, \quad a_2 = a_3 = -(1 \pm \sqrt{2})/2.$$

The resulting lowpass filter $G_0(\boldsymbol{z}) = A(\boldsymbol{z})/B(\boldsymbol{z})$ is given as

$$\begin{aligned} A(z_1, z_2) &= (3 - 2\sqrt{2}) + (8 - 8\sqrt{2})z_1 z_2 + (8\sqrt{2} - 12)z_1 z_2^3 \\ &+ (20\sqrt{2} - 14)z_1^2 z_2^2 + (8\sqrt{2} - 12)z_1^3 z_2 + (16\sqrt{2} - 16)z_1^2 z_2^3 \\ &+ (16\sqrt{2} - 16)z_1^3 z_2^2 + (8\sqrt{2} - 8)z_1^3 z_2^3 + (3 - 2\sqrt{2})z_1^4 z_2^4, \\ B(z_1, z_2) &= (2\sqrt{2} - 3) + (12 - 8\sqrt{2})z_1 z_2^3 \\ &+ (46 - 20\sqrt{2})z_1^2 z_2^2 + (12 - 8\sqrt{2})z_1^3 z_2 + (2\sqrt{2} - 3)z_1^4 z_2^4. \end{aligned}$$

This is a new orthogonal filter, to the best of our knowledge. The filter has diamond-like support of frequency response as shown in Fig. 4. It is easy to verify numerically that $B(z_1, z_2) \neq 0$ for $|z_1| = |z_2| = 1$ and hence the filter is stable [18](pp. 189–197). Implementing this filter using a difference equation requires 13 additions and 8 multiplications per output sample.

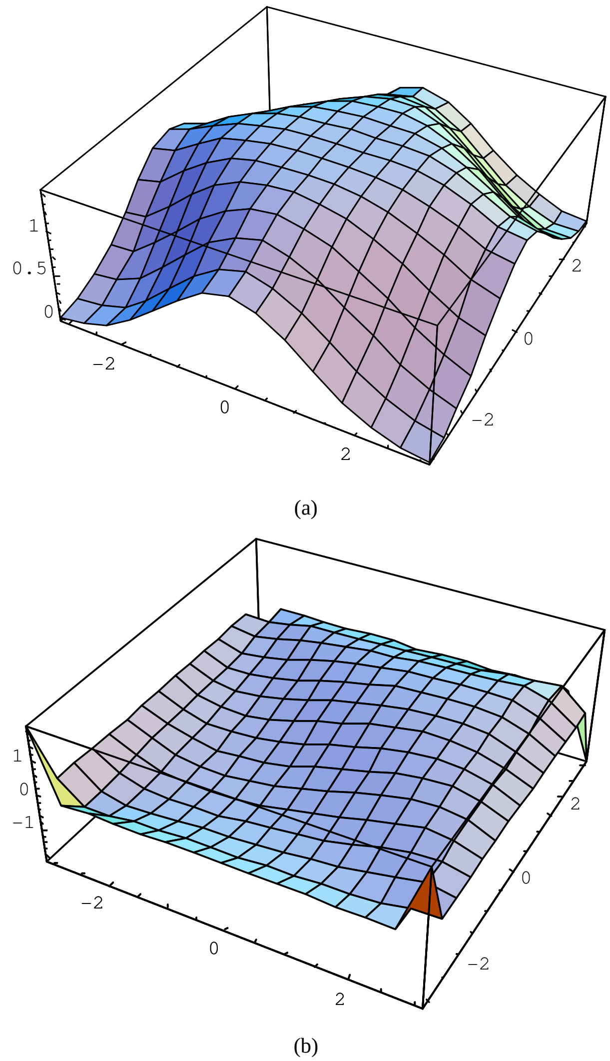


Fig. 4. Frequency responses of one orthogonal lowpass filter with second-order vanishing moment, obtained by the special paraunitary matrix and the Cayley transform. The design process is given in Example 4. (a) Magnitude; (b) Phase.

V. Conclusion

Designing multidimensional orthogonal filter banks amounts to designing paraunitary matrices. The paraunitary condition amounts to a set of nonlinear equations involving all matrix entries. We introduce special paraunitary matrices — paraunitary matrices with determinant 1. Since the last column of an $N \times N$ special paraunitary matrix is completely determined by its first $N - 1$ columns, the special paraunitary condition yields a smaller set of nonlinear equations. Thus, special paraunitary matrices have simpler structure than paraunitary matrices and are easier to design. Furthermore, since any paraunitary matrix can be characterized by a special paraunitary matrix and a phase factor, we can use special paraunitary matrices to simplify the design of paraunitary matrices and thus of multidimensional orthogonal filter banks.

The Cayley transform establishes a one-to-one mapping between special paraunitary matrices and special para-skew-Hermitian matrices and converts the nonlinear special paraunitary condition to the linear special para-skew-Hermitian condition. Using the Cayley transform to characterize special paraunitary matrices, we further simplify the characterization and design of multidimensional two-channel orthogonal filter banks. We propose the design process to impose vanishing moments on two-channel orthogonal filter banks. In our future work we will try to simplify characterization for orthogonal filter banks with more than two channels.

Appendix

A. Proof of Theorem 1

We first present a lemma before proving this theorem. In the following, for a matrix $\mathbf{A}$, the cofactor of $A_{i,j}$ is defined as $(-1)^{i+j} \det \mathbf{A}(i, j)$, where $\mathbf{A}(i, j)$ is the submatrix of $\mathbf{A}$ obtained by deleting its ith row and jth column.

Lemma 1: Suppose $\mathbf{U}(\boldsymbol{z})$ is an SPU matrix. Then $U_{i,j}(\boldsymbol{z})$ equals the cofactor of $U_{i,j}(\boldsymbol{z}^{-1})$.

Proof: Since $\mathbf{U}(\boldsymbol{z})$ is SPU, $\mathbf{U}^T(\boldsymbol{z}^{-1})\mathbf{U}(\boldsymbol{z}) = \mathbf{I}$ and $\det \mathbf{U}(\boldsymbol{z}) = 1$. Therefore,

$$\begin{aligned}\mathbf{U}^T(\boldsymbol{z}^{-1}) &= \mathbf{U}^{-1}(\boldsymbol{z}) \\ &= (\det \mathbf{U}(\boldsymbol{z}))^{-1} \cdot \text{adj}\, \mathbf{U}(\boldsymbol{z}) \\ &= \text{adj}\, \mathbf{U}(\boldsymbol{z}).\end{aligned}$$

Since the entry of adj $\mathbf{U}(\boldsymbol{z})$ at (i,j) is the cofactor of $U_{j,i}$, $U_{i,j}(\boldsymbol{z})$ equals the cofactor of $U_{i,j}(\boldsymbol{z}^{-1})$ for all i and j. ■

Now we are ready to prove Theorem 1. In the proof, we denote the jth column of $\mathbf{U}(\boldsymbol{z})$ by $U_j(\boldsymbol{z})$, while we denote its submatrix consisting of first $N-1$ columns by $\mathbf{U}_{N-1}(\boldsymbol{z})$.

For the necessary condition, suppose $\mathbf{U}(\boldsymbol{z})$ is SPU with the following decomposition:

$$\mathbf{U}(\boldsymbol{z}) = \begin{pmatrix} \mathbf{U}_{N-1}(\boldsymbol{z}) & U_N(\boldsymbol{z}) \end{pmatrix}.$$

Then $\mathbf{U}^T(\boldsymbol{z}^{-1})\mathbf{U}(\boldsymbol{z})$ becomes

$$\begin{pmatrix} \mathbf{U}_{N-1}^T(\boldsymbol{z}^{-1})\mathbf{U}_{N-1}(\boldsymbol{z}) & \mathbf{U}_{N-1}^T(\boldsymbol{z}^{-1})U_N(\boldsymbol{z}) \\ U_N^T(\boldsymbol{z}^{-1})\mathbf{U}_{N-1}(\boldsymbol{z}) & U_N^T(\boldsymbol{z}^{-1})U_N(\boldsymbol{z}) \end{pmatrix}. \tag{28}$$

Since $\mathbf{U}^T(\boldsymbol{z}^{-1})\mathbf{U}(\boldsymbol{z}) = \mathbf{I}_N$, (28) implies (4). The condition on $U_{N,j}(\boldsymbol{z})$ comes from Lemma 1 directly.

For the sufficient condition, we need prove that the given $\mathbf{U}(\boldsymbol{z})$ has determinant 1 and satisfies (1).

We first prove the determinant condition. For convenience, denote the cofactor and unsigned cofactor of $U_{i,j}(\boldsymbol{z})$ by $C_{i,j}(\boldsymbol{z})$ and $D_{i,j}(\boldsymbol{z})$ respectively. Moreover,

$$C_{i,j}(\boldsymbol{z}) = (-1)^{i+j} D_{i,j}(\boldsymbol{z}).$$

By assumption,

$$U_{i,N}(\boldsymbol{z}) = C_{i,N}(\boldsymbol{z}^{-1}) \quad \text{for } i = 1, \ldots, N.$$

The determinant of $\mathbf{U}(\boldsymbol{z})$ can be written as

$$\begin{aligned}\det \mathbf{U}(\boldsymbol{z}) &= \sum_{i=1}^{N} U_{i,N}(\boldsymbol{z}) C_{i,N}(\boldsymbol{z}) \\ &= \sum_{i=1}^{N} C_{i,N}(\boldsymbol{z}^{-1}) C_{i,N}(\boldsymbol{z}) \\ &= \sum_{i=1}^{N} D_{i,N}(\boldsymbol{z}^{-1}) D_{i,N}(\boldsymbol{z}).\end{aligned} \tag{29}$$

The $(N-1)$th compound matrix [19](pp. 19–20) of $N \times (N-1)$ matrix $\mathbf{U}_{N-1}(\boldsymbol{z})$, comp $\left(\mathbf{U}_{N-1}(\boldsymbol{z})\right)$ is an $N \times 1$ matrix and can be written as

$$\begin{pmatrix} D_{N,N}(\boldsymbol{z}), D_{N-1,N}(\boldsymbol{z}), \ldots, D_{1,N}(\boldsymbol{z}) \end{pmatrix}^T.$$

Similarly, the $(N-1)$th compound matrix of $(N-1) \times N$ matrix $\mathbf{U}_{N-1}^T(\boldsymbol{z}^{-1})$, comp $\left(\mathbf{U}_{N-1}^T(\boldsymbol{z}^{-1})\right)$ is a $1 \times N$ matrix and can be written as

$$\begin{pmatrix} D_{N,N}(\boldsymbol{z}^{-1}), D_{N-1,N}(\boldsymbol{z}^{-1}), \ldots, D_{1,N}(\boldsymbol{z}^{-1}) \end{pmatrix}.$$

Therefore,

$$\det \mathbf{U}(\boldsymbol{z}) = \text{comp}\left(\mathbf{U}_{N-1}^T(\boldsymbol{z}^{-1})\right) \text{comp}\left(\mathbf{U}_{N-1}(\boldsymbol{z})\right).$$

For any two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, comp $(\boldsymbol{A})$ comp $(\boldsymbol{B})$ = comp $(\boldsymbol{AB})$ [19](pp. 20). Therefore,

$$\begin{aligned}\det \mathbf{U}(\boldsymbol{z}) &= \text{comp}\left(\mathbf{U}_{N-1}^T(\boldsymbol{z}^{-1})\mathbf{U}_{N-1}(\boldsymbol{z})\right) \\ &= \text{comp}\left(\mathbf{I}_{N-1}\right) = 1.\end{aligned}$$

Now we prove that $\mathbf{U}(\boldsymbol{z})$ satisfies (1). By (28), it suffices to prove that

$$U_N^T(\boldsymbol{z}^{-1})U_N(\boldsymbol{z}) = 1, \tag{30}$$

$$\mathbf{U}_{N-1}^T(\boldsymbol{z}^{-1})U_N(\boldsymbol{z}) = 0. \tag{31}$$

To prove (30), we have

$$\begin{aligned}U_N^T(\boldsymbol{z}^{-1})U_N(\boldsymbol{z}) &= \sum_{i=1}^{N} U_{i,N}(\boldsymbol{z}) U_{i,N}(\boldsymbol{z}^{-1}) \\ &= \sum_{i=1}^{N} U_{i,N}(\boldsymbol{z}) C_{i,N}(\boldsymbol{z}),\end{aligned}$$

and because of (29), the last sum equals $\det \mathbf{U}(\boldsymbol{z}) = 1$.

To prove (31), it suffices to prove that

$$U_j^T(\boldsymbol{z})U_N(\boldsymbol{z}^{-1}) = 0, \quad \text{for } j = 1, \ldots, N-1. \tag{32}$$

For each j, let $\mathbf{W}(\boldsymbol{z})$ be an $N \times N$ matrix with

$$\mathbf{W}(\boldsymbol{z}) = (\, U_1(\boldsymbol{z}), U_2(\boldsymbol{z}), \ldots, U_{N-1}(\boldsymbol{z}), U_j(\boldsymbol{z}) \,).$$

Since the columns of $\mathbf{W}(\boldsymbol{z})$ are linearly dependent, the determinant of $\mathbf{W}(\boldsymbol{z})$ is 0. At the same time,

$$\begin{aligned}\det \mathbf{W}(\boldsymbol{z}) &= \sum_{i=1}^{N} U_{i,j}(\boldsymbol{z}) C_{i,N}(\boldsymbol{z}) \\ &= \sum_{i=1}^{N} U_{i,j}(\boldsymbol{z}) U_{i,N}(\boldsymbol{z}^{-1}) \\ &= U_j^T(\boldsymbol{z}) U_N(\boldsymbol{z}^{-1}),\end{aligned}$$

yielding (31). This completes the proof. ■

B. Proof of Proposition 6

By Condition (3) in Theorem 3,

$$2\, D(\boldsymbol{z}) = \det\left(D(\boldsymbol{z})\,\mathbf{I} + \mathbf{H}'(\boldsymbol{z})\right).$$

Since $\mathbf{H}'(\boldsymbol{z})$ is SPSH, tr $(\mathbf{H}'(\boldsymbol{z})) = 0$. Then by (22)

$$2\, D(\boldsymbol{z}) = D^2(\boldsymbol{z}) + \det \mathbf{H}'(\boldsymbol{z}),$$

which leads to

$$D(\boldsymbol{z}) = 1 \pm \left(1 - \det \mathbf{H}'(\boldsymbol{z})\right)^{1/2}.$$

For the two-channel case, (18) in Theorem 3 becomes

$$\begin{aligned}\mathbf{U}(\boldsymbol{z}) &= \text{adj}\left(D(\boldsymbol{z})\,\mathbf{I} + \mathbf{H}'(\boldsymbol{z})\right) - \mathbf{I} \\ &= D(\boldsymbol{z})\,\mathbf{I} + \text{adj}\, \mathbf{H}'(\boldsymbol{z}) - \mathbf{I}.\end{aligned}$$

Since $\mathbf{H}'(\boldsymbol{z})$ is a 2×2 SPSH matrix, $H'_{11}(\boldsymbol{z}) = -H'_{00}(\boldsymbol{z})$, and thus adj $\mathbf{H}'(\boldsymbol{z}) = -\mathbf{H}'(\boldsymbol{z})$. Therefore,

$$\begin{aligned}\mathbf{U}(\boldsymbol{z}) &= \left(D(\boldsymbol{z}) - 1\right)\mathbf{I} - \mathbf{H}'(\boldsymbol{z}) \\ &= \pm\left(1 - \det \mathbf{H}'(\boldsymbol{z})\right)^{1/2}\mathbf{I} - \mathbf{H}'(\boldsymbol{z}).\end{aligned}$$

■

REFERENCES

[1] G. Karlsson and M. Vetterli, "Theory of two-dimensional multirate filter banks," *IEEE Trans. Acoust., Speech, and Signal Proc.*, vol. 38, no. 6, pp. 925–937, June 1990.

[2] E. Viscito and J. P. Allebach, "The analysis and design of multidimensional FIR perfect reconstruction filter banks for arbitrary sampling lattices," *IEEE Trans. Circ. and Syst.*, vol. 38, no. 1, pp. 29–42, January 1991.

[3] J. Kovačević and M. Vetterli, "Nonseparable multidimensional perfect reconstruction filter banks and wavelet bases for $\mathcal{R}^n$," *IEEE Transactions on Information Theory, Special Issue on Wavelet Transforms and Multiresolution Signal Analysis*, vol. 38, no. 2, pp. 533–555, March 1992.

[4] K. Gröchenig and W. R. Madych, "Multiresolution analysis, Haar bases and self-similar tilings of R^n," *IEEE Transactions on Information Theory, Special Issue on Wavelet Transforms and Multiresolution Signal Analysis*, vol. 38, no. 2, pp. 556–568, March 1992.

[5] R. H. Bamberger and M. J. T. Smith, "A filter bank for the directional decomposition of images: Theory and design," *IEEE Trans. Signal Proc.*, vol. 40, no. 4, pp. 882–893, April 1992.

[6] T. Chen and P. P. Vaidyanathan, "Recent developments in multidimensional multirate systems," *IEEE Trans. on CSVT*, vol. 3, no. 2, pp. 116–137, April 1993.

[7] D. B. H. Tay and N. G. Kingsbury, "Flexible design of multidimensional perfect reconstruction FIR 2-band filters using transformations of variables," *IEEE Trans. Image Proc.*, vol. 2, no. 4, pp. 466–480, October 1993.

[8] P. P. Vaidyanathan, *Multirate Systems and Filter Banks*. Prentice-Hall, 1993.

[9] M. Vetterli and J. Kovačević, *Wavelets and Subband Coding*. Prentice-Hall, 1995.

[10] M. N. Do and M. Vetterli, "The contourlet transform: An efficient directional multiresolution image representation," *IEEE Trans. Image Proc.*, to appear, `http://www.ifp.uiuc.edu/~minhdo/publications`.

[11] I. Daubechies, "Orthonormal bases of compactly supported wavelets," *Commun. on Pure and Appl. Math.*, vol. 41, pp. 909–996, November 1988.

[12] A. Cohen and I. Daubechies, "Nonseparable bidimensional wavelet bases," *Rev. Mat. Iberoamericana*, vol. 9, no. 1, pp. 51–137, 1993.

[13] S. Basu and A. Fettweis, "On synthesizable multidimensional loseless two-ports," *IEEE Trans. Circ. and Syst.*, vol. 35, no. 12, pp. 1478–1486, December 1988.

[14] J. Zhou, M. N. Do, and J. Kovačević, "Multidimensional orthogonal filter bank characterization and design using the Cayley transform," *IEEE Trans. Image Proc.*, vol. 14, no. 6, June 2005.

[15] C. Herley and M. Vetterli, "Wavelets and recursive filter banks," *IEEE Trans. Signal Proc.*, vol. 41, no. 8, pp. 2536–2556, August 1993.

[16] A. V. Oppenheim, R. W. Schafer, and J. R. Buck, *Discrete-Time Signal Processing*. Prentice-Hall, 1999.

[17] N. K. Bose and A. Fettweis, "Skew symmetry and orthogonality in the equivalent representation problem of a time-varying multiport inductor," *IEEE Trans. Circ. and Syst.*, vol. 51, no. 7, pp. 1321–1329, July 2004.

[18] D. E. Dudgeon and R. M. Mersereau, *Multidimensional Digital Signal Processing*. Prentice-Hall, 1984.

[19] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge University Press, 1999.

Jianping Zhou received the B.Sc. degree in applied mathematics from the Xi'an Jiaotong University, China, in 1997, and the M.Eng. degree in signal and information processing from the Peking University, China, in 2000.

He is currently a Ph.D. candidate in Department of Electrical and Computer Engineering and a Research Assistant with the Coordinated Science Laboratory, University of Illinois at Urbana-Champaign. From 2000 to 2002, he was an Assistant Researcher in the Internet Media Group, and then Wireless and Networking Group, Microsoft Research Asia, China. His research interests include wavelets, computer vision, image and multidimensional signal processing, multimedia coding and communication, multimedia delivery over wireless Internet.

Minh N. Do (M'02) was born in Thanh Hoa, Vietnam, in 1974. He received the B.Eng. degree in computer engineering from the University of Canberra, Australia, in 1997, and the Dr.Sci. degree in communication systems from the Swiss Federal Institute of Technology Lausanne (EPFL), Switzerland, in 2001.

Since 2002, he has been an Assistant Professor with the Department of Electrical and Computer Engineering and a Research Assistant Professor with the Coordinated Science Laboratory and the Beckman Institute, University of Illinois at Urbana-Champaign. His research interests include wavelets, image and multidimensional signal processing, multiscale geometric analysis, and visual information representation.

Dr. Do received a Silver Medal from the 32nd International Mathematical Olympiad in 1991, a University Medal from the University of Canberra in 1997, the best doctoral thesis award from the Swiss Federal Institute of Technology Lausanne in 2001, and a CAREER award from the National Science Foundation in 2003.

Jelena Kovačević (S'88-M'91-SM'96-F'02) received the Dipl. Electr. Eng. degree from the Electrical Engineering Department, University of Belgrade, Yugoslavia, in 1986, and the Master of Science and Ph.D. degrees from Columbia University, New York, NY, in 1988 and 1991, respectively.

From 1987 to 1991, she was a Graduate Research Assistant at Columbia University. In the summer of 1985, she worked for Gaz de France, Paris, France, during the summer of 1987, for INTELSAT, Washington, D.C., and in the summer of 1988, for Pacific Bell, San Ramon, CA. In the fall of 1986, she was a Teaching Assistant at the University of Belgrade. From 1991-2002, she was with Bell Labs, Murray Hill, NJ, as a Member of Technical Staff. She is a co-founder and Technical VP of xWaveforms, based in New York City, NY. In 2003, she joined the Departments of Biomedical Engineering & Electrical and Computer Engineering at Carnegie Mellon University. She was also an Adjunct Professor at Columbia University. Her research interests include wavelets, frames, biomedical signal processing, multirate signal processing, data compression and signal processing for communications. She is a coauthor on the paper for which Aleksandra Mojsilovic received the Young Author Best Paper Award. She is a coauthor (with Martin Vetterli) of the book "Wavelets and Subband Coding" (Englewood Cliffs, NJ: Prentice Hall, 1995). She is on the Editorial Boards of the Journal of Applied and Computational Harmonic Analysis, Journal of Fourier Analysis and Applications and the Signal Processing Magazine.

Dr. Kovačević received the Belgrade October Prize, highest Belgrade prize for student scientific achievements awarded for the Engineering Diploma Thesis in October 1986 and the E.I. Jury Award at Columbia University for outstanding achievement as a graduate student in the areas of systems, communication or signal processing. She is the Fellow of the IEEE and the Editor-in-Chief of the IEEE Transactions on Image Processing. She served as an Associate Editor of the IEEE Transactions on Signal Processing, as a Guest Co-Editor (with Ingrid Daubechies) of the Special Issue on Wavelets of the Proceedings of the IEEE, and as a Guest Co-Editor (with Martin Vetterli) of the Special Issue on Transform Coding of the Signal Processing Magazine. She is a Member-at-Large of the IEEE Signal Processing Society Board of Governors and serves on the IMDSP Technical Committee of the IEEE Signal Processing Society. She was the Co-Chair (with Vivek Goyal) of the DIMACS Workshop on Source Coding and Harmonic Analysis and a General Co-Chair (with Jan Allebach) of the Ninth Workshop on Image and Multidimensional Signal Processing.