Multidimensional Filter Bank Signal Reconstruction From Multichannel Acquisition

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***Abstract*—We study the theory and algorithms of an optimal use of multidimensional signal reconstruction from multichannel acquisition by using a filter bank setup. Suppose that we have an N-channel convolution system, referred to as N analysis filters, in M dimensions. Instead of taking all the data and applying multichannel deconvolution, we first reduce the collected data set by an integer $M \times M$ uniform sampling matrix $\boldsymbol{D}$, and then search for a synthesis polyphase matrix which could perfectly reconstruct any input discrete signal. First, we determine the existence of perfect reconstruction (PR) systems for a given set of finite impulse response (FIR) analysis filters. Second, we present an efficient algorithm to find a sampling matrix with maximum sampling rate and to find a FIR PR synthesis polyphase matrix for a given set of FIR analysis filters. Finally, once a particular FIR PR synthesis polyphase matrix is found, we can characterize all FIR PR synthesis matrices, and then find an optimal one according to design criteria including robust reconstruction in the presence of noise.**

***Index Terms*—Filter bank, FIR multidimensional filters, multichannel acquisition, multidimensional sampling, perfect reconstruction.**

I. Introduction

In recent decades, multirate systems have played an increasingly important role in engineering area. The ideas of multirate systems have been extended from one dimensional systems to multidimensional systems; see [2], [3]. For multichannel acquisition applications, the most commonly employed multirate systems is N-channel perfect reconstruction (PR) finite impulse response (FIR) uniform filter banks [4], [5], [6], [7].

The reconstruction of a signal from its filtered and downsampled versions was discussed by Papoulis [8]. Using the generalized sampling expansion of Papoulis, the input signal can be uniquely determined in terms of the samples from the data set $\{g_k(nT)\}_{k=1,\dots,m}$ of the response $g_k(t)$ where each filter output is sampled at $(1/m)$th the Nyquist rate. It can be extended to multidimensional version [9], [10], [11]. For digital signal processing, the generalized sampling expansion is a special case of perfect reconstruction filter banks.

In the traditional setting of the perfect reconstruction filter banks, the analysis filters and the sampling matrix are both given (or estimated) in the uniform filter bank system. The goal is to find synthesis filters such that the system remains a perfect reconstruction for any input discrete signal [12], [13], [14], [15], [16], [17]. In this paper, we consider that the analysis filters are given, but the sampling matrix is unknown. Now, our new goal is to find a suitable sampling matrix and a synthesis polyphase matrix, which satisfies the perfect reconstruction condition. An application of this setting arises when suppose that we have an N-channel convolution system in M dimensions. Instead of taking all the data and applying multichannel deconvolution, we can first reduce the collected data set by an integer $M \times M$ sampling matrix $\boldsymbol{D}$ and then search for a PR synthesis polyphase matrix (see Fig. 1). Of course, we want a sampling factor of $\boldsymbol{D}$ to be as large as possible, because it would give us a minimum collected data set. To address this problem, we have to answer the following questions:

Question 1: Given a set of analysis filters, can we have a PR system with some sampling matrices $\boldsymbol{D}$ and some FIR synthesis polyphase matrices?

Question 2: If so, find a sampling matrix $\boldsymbol{D}$ having a maximal sampling factor and a particular FIR PR synthesis polyphase matrix?

Question 3: Among all maximum density sampling matrices and all FIR PR synthesis polyphase matrices, can we find an optimal solution which provides a robust reconstruction in the presence of noise?

These problems can be approached differently in one dimensional (1D) and multidimensional (MD) cases. In 1D, since $\boldsymbol{D}$ is a scalar, we just need to do a linear search on sampling factors. However, in MD, there are infinitely many matrices with a given sampling factor. We will address it by using the Hermite normal form, and the Smith normal form.

The rest of the paper is organized as follows. In Section II, we state the general problem and provide our main algorithm in a flowchart. Then we discuss each component in the corresponding section. In Section III, we have a review on polyphase representation and provide an algorithm to obtain the analysis polyphase matrix. In Section IV, we summarize the inverse (resp.: Laurent polynomial) polynomial matrix problem from our previous paper [18]. In Section V, we provide a method to represent the sampling matrices up to equivalence classes. In Section VI, we provide necessary condition for a PR system with some sampling matrices and some Laurent polynomial synthesis polyphase matrices. In Section VI, we provide a method to search for a maximum density sampling matrix and compute the worst case iteration of our main algorithm. Finally, we discuss the optimizations of Laurent polynomial PR synthesis polyphase matrices ac-

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Some preliminary results of this paper have been presented at [1].

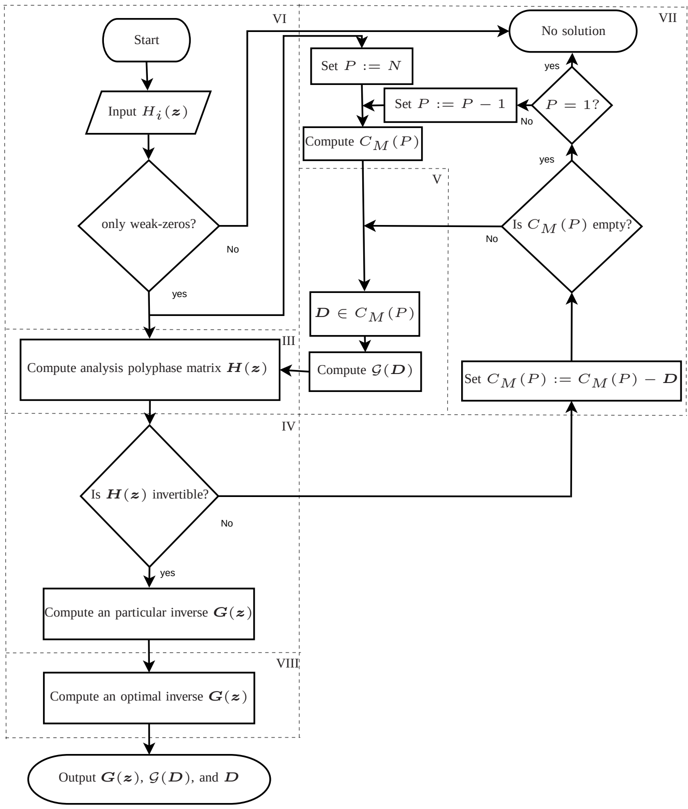

Fig. 2. Flowchart of searching for a maximal sampling rate matrix and an optimal PR synthesis polyphase matrix.

cording to some design criteria in Section VIII and make a conclusion in Section IX.

Notation: Let M and N be the dimension of the signals, and the number of channels, respectively. Let $\boldsymbol{D}$ be an $M \times M$ sampling integer matrix where $|\det \boldsymbol{D}|$ is nonzero. Let P be the sampling factor at each channel, $P := |\det \boldsymbol{D}|$. Let $\boldsymbol{z}$ be an M-dimensional complex variable $\boldsymbol{z} = (z_1, ..., z_M)$ in $\mathbb{C}^M$. Let $\boldsymbol{n} = (n_1, ..., n_M) \in \mathbb{Z}^M$. We define $\boldsymbol{z}^{\boldsymbol{n}} = \prod_{i=1}^{M} z_i^{n_i}$. Similarly, we define $\boldsymbol{z}^{\boldsymbol{D}} = (\boldsymbol{z}^{\boldsymbol{d}_1}, \boldsymbol{z}^{\boldsymbol{d}_2}, ..., \boldsymbol{z}^{\boldsymbol{d}_M})$, where $\boldsymbol{d}_i$ is the ith column of $\boldsymbol{D}$. Let LAT($\boldsymbol{D}$) be the set of all vectors of the form $\boldsymbol{Dm}$ for all $\boldsymbol{m} \in \mathbb{Z}^M$. In this paper, we assume all signals and filters are with finite supports. As shown in Fig. 1(a), let $x, \hat{x}, \boldsymbol{h} = (h_0, h_1, ..., h_{N-1}), \boldsymbol{y} = (y_0, y_1, ..., y_{N-1})$, and $\boldsymbol{g} = (g_0, g_1, ..., g_{N-1})$ be the input discrete signal, output discrete signal, analysis filters, sub-band signals, and synthesis filters, respectively. Suppose f is a signal or filter, then $f[\boldsymbol{n}]$ is a value evaluated at $\boldsymbol{n}$. A multivariate Laurent polynomial $p(\boldsymbol{z})$ is an expression of the form $p(\boldsymbol{z}) = \sum_{i=m_1}^{m_2} a_i \boldsymbol{z}^{\boldsymbol{n}_i}$ where $a_i \in \mathbb{C}$, $\boldsymbol{n}_i \in \mathbb{Z}^M$, and $m_1, m_2 \in \mathbb{Z}$. The z-transform of an FIR filter f can be expressed as a Laurent polynomial $F(\boldsymbol{z}) = \sum_{\boldsymbol{n} \in \mathbb{Z}^M} f[\boldsymbol{n}] \boldsymbol{z}^{-\boldsymbol{n}}$. Let $X(\boldsymbol{z}), \hat{X}(\boldsymbol{z}), H_i(\boldsymbol{z}), Y_i(\boldsymbol{z})$, and $G_i(\boldsymbol{z})$ be the z-transform of the input signal, output signal, ith analysis filter, ith sub-band signal, and ith synthesis filter, respectively. We denote matrices by boldface letters. Let $\boldsymbol{X}(\boldsymbol{z}), \hat{\boldsymbol{X}}(\boldsymbol{z}), \boldsymbol{H}(\boldsymbol{z}), \boldsymbol{Y}(\boldsymbol{z})$, and $\boldsymbol{G}(\boldsymbol{z})$ be the polyphase representation matrices of the input signal, output signal, analysis filters, sub-band signals, and synthesis filters, respectively. We denote $M_{i,j}(\boldsymbol{z})$ be the (i, j) entry of $\boldsymbol{M}(\boldsymbol{z})$.

II. Main Algorithm

The data acquisition can be shown in Fig. 1(a). We consider the polynomial analysis filters $H_0(\boldsymbol{z})$, $H_2(\boldsymbol{z})$,...,$H_{N-1}(\boldsymbol{z})$ to be given (i.e., point spread function of sampling devices), while the sampling matrix $\boldsymbol{D}$ is unknown. We provide an efficient algorithm to find a sampling matrix $\boldsymbol{D}$ with a maximal sampling rate and an optimal Laurent polynomial PR synthesis polyphase matrix $\boldsymbol{G}(\boldsymbol{z})$. This is a generalization of the multichannel deconvolution problem (i.e., set $\boldsymbol{D} = \boldsymbol{I}$). The flowchart in Fig. 2 illustrates our algorithm. We label each component in the flowchart at the right hand corner. We will discuss each component in the corresponding section.

III. Polyphase Decomposition

In this section, we discuss how to compute the analysis polyphase decomposition matrix $\boldsymbol{H}(\boldsymbol{z})$ from the analysis filters and the group of representatives $\mathcal{G}(\boldsymbol{D})$. First, we briefly review the relationship between the polyphase matrices and the filters.

In [2], Vaidyanathan defined the set of representatives as $\mathcal{N}(\boldsymbol{D}) = \{\boldsymbol{Dt} \in \mathbb{Z}^M \mid \boldsymbol{t} \in [0, 1)^M\}$. In instead of using this definition, we define the group of representatives $(\mathcal{G}(\boldsymbol{D}), +)$ to be the quotient group of $\boldsymbol{D}\mathbb{Z}^M$ in $\mathbb{Z}^M$ (i.e., $\mathbb{Z}^M / \boldsymbol{D}\mathbb{Z}^M$) together with the addition operation. The first observation is that both $\mathcal{N}(\boldsymbol{D})$ and $\mathcal{G}(\boldsymbol{D})$ have the same size. The second observation is that a set of representatives of $\mathcal{G}(\boldsymbol{D})$ is not unique. Let $\{[\boldsymbol{l}_0]_D, [\boldsymbol{l}_1]_D, ..., [\boldsymbol{l}_{P-1}]_D\}$ and $\{[\tilde{\boldsymbol{l}}_0]_D, [\tilde{\boldsymbol{l}}_1]_D, ..., [\tilde{\boldsymbol{l}}_{P-1}]_D\}$ be two sets of the representatives of $\mathcal{G}(\boldsymbol{D})$ where $[\boldsymbol{a}_j]_D$ denotes the coset $\{\boldsymbol{a}_j + \boldsymbol{Dt} \mid \boldsymbol{t} \in \mathbb{Z}^M\}$. Then, for every i, there exists $\boldsymbol{t}_i$ such that $\boldsymbol{l}_i - \tilde{\boldsymbol{l}}_{\rho(i)} = \boldsymbol{Dt}_i$ where ρ is a permutation. In Section V, we will discuss how to generate a set of representatives of $\mathcal{G}(\boldsymbol{D})$ for a given $\boldsymbol{D}$ systematically.

Let $\{[\boldsymbol{l}_0]_D, ..., [\boldsymbol{l}_{P-1}]_D\}$ be a set of representatives of $\mathcal{G}(\boldsymbol{D})$. By the polyphase representation (PD) [2], [3], the $P \times 1$ polyphase matrix of the input signal is given by $\boldsymbol{X}(\boldsymbol{z}) = (X_0(\boldsymbol{z}), ..., X_{(P-1)}(\boldsymbol{z}))^T$ where the jth entry is a Laurent polynomial

$$X_j(\boldsymbol{z}) = \sum_{\boldsymbol{m} \in \mathbb{Z}^M} x[\boldsymbol{Dm} + \boldsymbol{l}_j] \boldsymbol{z}^{-\boldsymbol{m}} \tag{1}$$

and its z-transform can be expressed as a Laurent polynomial

$$X(\boldsymbol{z}) = \sum_{\boldsymbol{n} \in \mathbb{Z}^M} x[\boldsymbol{n}] \boldsymbol{z}^{-\boldsymbol{n}} = \sum_{[\boldsymbol{l}_j]_D \in \mathcal{G}(\boldsymbol{D})} X_j(\boldsymbol{z}^{\boldsymbol{D}}) \boldsymbol{z}^{-\boldsymbol{l}_j}. \tag{2}$$

Note: For the sake of algorithmic and programming purpose (i.e., the efficiency and compactness of storage and computation of the matrix), we prefer to define the polyphase component in terms of the sampling matrix $\boldsymbol{D}$ and $\mathcal{G}(\boldsymbol{D})$ instead of the lattice and a coset of the lattice. Readers, who are interested in the general definition (i.e., lattice form) of the polyphase representation, should refer to [19].

By the polyphase representation, the analysis polyphase matrix $\boldsymbol{H}(\boldsymbol{z})$ shown in Fig. 1(b) is defined as the $N \times P$ matrix $\boldsymbol{H}(\boldsymbol{z}) = [H_{i,j}(\boldsymbol{z}))]_{i=0,...,N-1;\, j=0,...,P-1}$ where the (i, j) entry is a Laurent polynomial $H_{i,j}(\boldsymbol{z}) = \sum_{\boldsymbol{m} \in \mathbb{Z}^M} h_i[\boldsymbol{Dm} -$

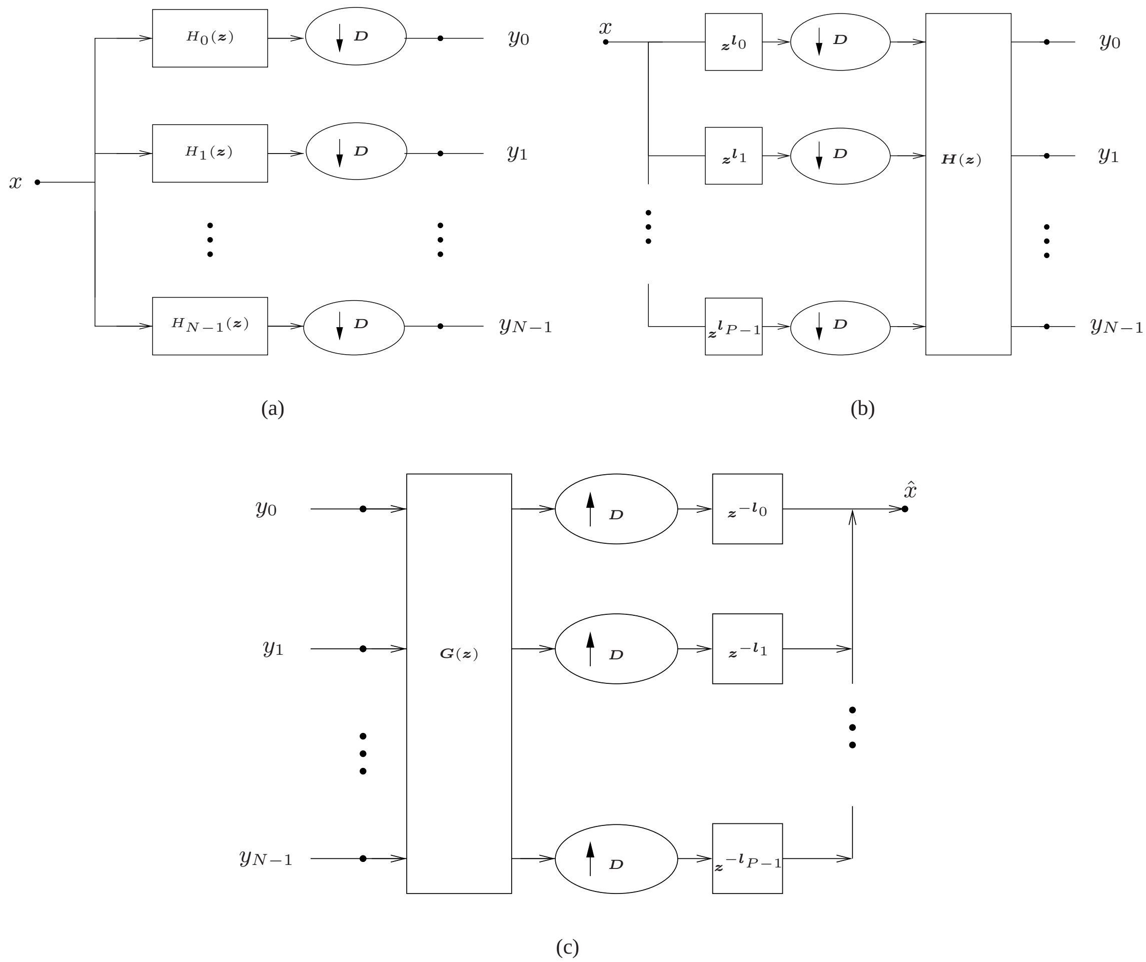

Fig. 1. (a) Signal acquisition or analysis part: Multichannel convolution followed downsampling by $\boldsymbol{D}$. (b) Polyphase representation of the analysis part. (c) Synthesis polyphase reconstruction.

$\boldsymbol{l}_j]\boldsymbol{z}^{-\boldsymbol{m}}$. Then the z-transform of the ith analysis filter can be described as

$$H_i(\boldsymbol{z}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^M} h_i[\boldsymbol{n}]\boldsymbol{z}^{-\boldsymbol{n}} = \sum_{[\boldsymbol{l}_j]_D\in\mathcal{G}(\boldsymbol{D})} \boldsymbol{z}^{\boldsymbol{l}_j} H_{i,j}(\boldsymbol{z}^{\boldsymbol{D}}). \quad (3)$$

Now we can express the $N\times 1$ polyphase matrix of sub-band signals shown in Fig. 1(b) as $\boldsymbol{Y}(\boldsymbol{z}) = \boldsymbol{H}(\boldsymbol{z})\boldsymbol{X}(\boldsymbol{z})$ where $\boldsymbol{Y}(\boldsymbol{z}) = (Y_0(\boldsymbol{z}), ..., Y_{N-1}(\boldsymbol{z}))^T$ with its kth entry $Y_k(\boldsymbol{z}) = \sum_{\boldsymbol{m}\in\mathbb{Z}^M} y_k[\boldsymbol{m}]z^{-\boldsymbol{m}}$. Similarly, on the synthesis side, the $P\times 1$ polyphase matrix of output signal can be expressed as $\hat{\boldsymbol{X}}(\boldsymbol{z}) = \boldsymbol{G}(\boldsymbol{z})\boldsymbol{Y}(\boldsymbol{z})$ where $\hat{\boldsymbol{X}}(\boldsymbol{z}) = (\hat{X}_0(\boldsymbol{z}), ..., \hat{X}_{P-1}(\boldsymbol{z}))^T$ with its ith entry $\hat{X}_i(\boldsymbol{z}) = \sum_{\boldsymbol{m}\in\mathbb{Z}^M} \hat{x}[\boldsymbol{m}\boldsymbol{D} + \boldsymbol{l}_i]z^{-\boldsymbol{m}}$ and $\boldsymbol{G}(\boldsymbol{z}) = [G_{j,i}(\boldsymbol{z})]_{i=0,...,P-1;\, j=0,...,N-1}$ is the synthesis polyphase matrix shown in Fig. 1(c) which corresponds to the synthesis filters $G_i(\boldsymbol{z}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^M} g_i[\boldsymbol{n}]\boldsymbol{z}^{-\boldsymbol{n}} = \sum_{[\boldsymbol{l}_j]_D\in\mathcal{G}(\boldsymbol{D})} \boldsymbol{z}^{-\boldsymbol{l}_j} G_{j,i}(\boldsymbol{z}^{\boldsymbol{D}})$ in filter banks. We say it is a perfect reconstruction (PR) system if the output signal equals to the input signal for every input (i.e., $\boldsymbol{X}(\boldsymbol{z}) = \hat{\boldsymbol{X}}(\boldsymbol{z})$). The above discussion shows that $\hat{\boldsymbol{X}}(\boldsymbol{z}) = \boldsymbol{G}(\boldsymbol{z})\boldsymbol{Y}(\boldsymbol{z}) = \boldsymbol{G}(\boldsymbol{z})\boldsymbol{H}(\boldsymbol{z})\boldsymbol{X}(\boldsymbol{z})$. This means that the PR condition holds if and only if $\boldsymbol{G}(\boldsymbol{z})\boldsymbol{H}(\boldsymbol{z}) = \boldsymbol{I}$.

Remark 1: According to the theory of filter banks, the choice of analysis filters, synthesis filters, and uniform sampling matrix is an intrinsic property of the PR condition. It is independent to the choice of representatives of $\mathcal{G}(\boldsymbol{D})$.

Following the above discussion, for completeness we provide an algorithm to compute the analysis polyphase matrix.

Algorithm 1 Analysis Polyphase Matrix Computation

The computational algorithm for the analysis polyphase matrix
Input: $\{H_0(\boldsymbol{z}), ..., H_{N-1}(\boldsymbol{z})\}$, $\{[\boldsymbol{l}_0]_D, ..., [\boldsymbol{l}_{P-1}]_D\}$, and $\boldsymbol{D}$
Output: $N\times P$ polyphase matrix $\boldsymbol{H}(\boldsymbol{z})$

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set H_{i,j}(z) = 0 for all i = 0,...,N-1, j = 0,...,P-1
for i = 0 to N-1 do
    for each n in the support of H_i(z) (with h_i[n] as the corresponding filter coefficient) do
        for j = 0 to P-1 do
            w = D^{-1}(n - l_j)
            if w is an integer vector, then H_{i,j}(z) = H_{i,j}(z) + h_i[n] z^w
            end if
        end for
    end for
end for
output H(z)
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IV. Inverse Laurent Polynomial Matrix Problem

Suppose we now have an analysis polyphase matrix $\boldsymbol{H}(\boldsymbol{z})$. To check the PR condition for this given $\boldsymbol{H}(\boldsymbol{z})$, we only need to verify if $\boldsymbol{H}(\boldsymbol{z})$ is a left invertible matrix or not. In this section, we restate our previous results in [18], which we was concerned with the probability of the invertibility of Laurent polynomial matrix. First, we provide computable conditions to test the left invertibility. Second, we propose algorithms to find a particular inverse. For further details, please refer to our paper [18].

Definition 1 (Left Invertible): An $N \times P$ polynomial (resp.: Laurent polynomial) matrix $\boldsymbol{H}(\boldsymbol{z})$ is said to be *polynomial (resp.: Laurent polynomial) left invertible* if there exists a $P \times N$ polynomial (resp.: Laurent polynomial) matrix $\boldsymbol{G}(\boldsymbol{z})$ such that $\boldsymbol{G}(\boldsymbol{z})\boldsymbol{H}(\boldsymbol{z}) = \boldsymbol{I}$.

The following two propositions can help us determine if $\boldsymbol{H}(\boldsymbol{z})$ is (resp.: Laurent polynomial) polynomial left invertible or not.

Proposition 1: [18] Suppose $\boldsymbol{H}(\boldsymbol{z})$ is an $N \times P$ polynomial matrix. Let $S = \langle \boldsymbol{h}_0(\boldsymbol{z}), ..., \boldsymbol{h}_{N-1}(\boldsymbol{z})\rangle$ be the $\mathbb{C}[\boldsymbol{z}]$-submodule of $\mathbb{C}[\boldsymbol{z}]^P$ generated by the rows $\boldsymbol{h}_i(\boldsymbol{z})$ of $\boldsymbol{H}(\boldsymbol{z})$. Then $\boldsymbol{H}(\boldsymbol{z})$ is invertible if and only if the reduced Gröbner basis of S is $\{\boldsymbol{e}_i\}_{i=1,...,P}$ where $\boldsymbol{e}_i$ is the ith row of the $P \times P$ identity matrix. (For the detail of Gröbner basis and submodule, please refer to [20]).

Proposition 2: [18] Suppose $\boldsymbol{H}(\boldsymbol{z})$ is an $N \times P$ Laurent polynomial matrix. Consider the $(N+P) \times P$ matrix $\boldsymbol{H}'(\boldsymbol{z}, w) = \begin{pmatrix} \boldsymbol{z}^{\boldsymbol{m}}\boldsymbol{H}(\boldsymbol{z}) \\ (1 - z_1 z_2 ... z_M w)\boldsymbol{I} \end{pmatrix}$ such that $\boldsymbol{z}^{\boldsymbol{m}}\boldsymbol{H}(\boldsymbol{z})$ is a polynomial matrix, where $\boldsymbol{m} \in \mathbb{N}^M$, w is a new variable, and $\boldsymbol{I}$ is the $P \times P$ identity matrix. Then $\boldsymbol{H}(\boldsymbol{z})$ is Laurent polynomial left invertible if and only if $\boldsymbol{H}'(\boldsymbol{z}, w)$ is polynomial left invertible.

Based on these two propositions, we propose Algorithm 2 to compute a particular Laurent polynomial inverse.

Algorithm 2 [18] Particular Laurent Polynomial Inverse

The computational algorithm for a Laurent polynomial left inverse matrix
Input: $N \times P$ Laurent polynomial matrix $\boldsymbol{H}(\boldsymbol{z})$ with M variables
Output: $P \times N$ Laurent polynomial matrix $\boldsymbol{G}(\boldsymbol{z})$, if it exists

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multiply H(z) by a common monomial z^m
compute the reduced Gröbner basis of {h'_1(z,w), ..., h'_N(z,w)} where h'_i(z,w) is a row
    of H'(z,w) and the associated transformation matrix T(z,w)
if the reduced Gröbner basis is {e_i}_{i=1,..,P}, then output
    z^{-m}(T_ij(z, z^{-1}))_{i=1,...,P; j=1,...,N}
else there is no solution
end if
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V. Representation of Sampling Matrices

As we mentioned in the introduction, when the dimension of signal is greater than one, there are infinitely many sampling matrices $\boldsymbol{D}$ with the same sampling rate P. In this section, we employ the Hermite normal form to show that there is only a finite number of representative sampling matrices for the given P up to equivalence classes. Then we demonstrate how to generate the set of the Hermite normal forms. Next, by applying the Smith normal form, we can compute $\mathcal{G}(\boldsymbol{D})$, which we use as one of the inputs to compute the analysis polyphase matrix in Algorithm 1.

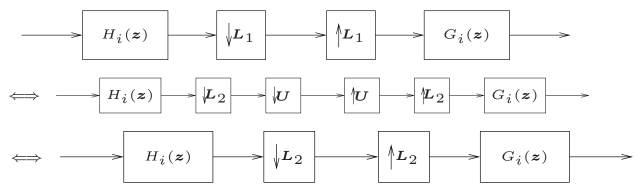


Fig. 3. Equivalent filters with sampling matrices $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ where $\boldsymbol{L}_2\boldsymbol{U} = \boldsymbol{L}_1$.

Proposition 3: [19] $\text{LAT}(\boldsymbol{L}_1) = \text{LAT}(\boldsymbol{L}_2)$ if and only if $\boldsymbol{L}_2\boldsymbol{U} = \boldsymbol{L}_1$, where $\boldsymbol{U}$ is unimodular integer matrix (i.e., $|\det \boldsymbol{U}| = 1$).

By Proposition 3 and Fig. 3, we know that the system with sampling matrix $\boldsymbol{L}_1$ has PR if and only if the system with sampling matrix $\boldsymbol{L}_2$ has PR. Though this proposition greatly reduced the search space, we still do not know whether the search space is finite or not. Moreover, it does not provide us any search method. We will address these problems by using the Hermite Normal Form.

Theorem 1 (Hermite Normal Form): [21] Given an $M \times M$ nonsingular integer-valued matrix $\boldsymbol{D}$, there exists an $M \times M$ unimodular integer matrix $\boldsymbol{U}$ such that $\boldsymbol{D}\boldsymbol{U} = \boldsymbol{E}$, the Hermite normal form of $\boldsymbol{D}$, whose entries satisfy

$$\begin{aligned} &E_{i,j} = 0, \quad \forall j > i, \\ &E_{i,i} > 0, \quad \forall i, \\ &E_{i,j} \leq 0 \text{ and } |E_{i,j}| < E_{i,i}, \quad \forall j < i. \end{aligned}$$

Furthermore, the Hermite normal form $\boldsymbol{E}$ of $\boldsymbol{D}$ is unique.

Remark 2: We know that $\prod_{i=1}^{M} E_{i,i} = \det(\boldsymbol{E}) = \det(\boldsymbol{D})\det(\boldsymbol{U}) = P$.

We define $C_M(P)$ to be the set of the $M \times M$ Hermite normal form matrices for a given absolute determinant P. By Theorem 1 and Remark 2, we can generate $C_M(P)$ systematically for a given P. Let us explain it through an example.

Example 1: Let $M = 2$ and $P = 4$. By remark 2, the candidate of the diagonal elements are $\{\{1,4\},\{2,2\},\{4,1\}\}$. Then, by Theorem 1, the complete set $C_2(4)$ of the representative sampling matrices with the sampling rate 4 is

$$\left\{ \begin{pmatrix} 1 & 0 \\ -3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Now let us consider the problem of generating a set of representatives of $\mathcal{G}(\boldsymbol{D})$. We employ the Smith normal form to address this problem.

Theorem 2 (Smith Normal Form): [21] Any nonsingular integer matrix $\boldsymbol{D}$ can always be decomposed as $\boldsymbol{D} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{V}$ where $\boldsymbol{U}$ and $\boldsymbol{V}$ are unimodular integer matrices and $\boldsymbol{\Lambda}$ is diagonal matrix with positive-integer elements such that the diagonal λ_i's of $\boldsymbol{\Lambda}$ satisfy that λ_i is a divisor of λ_{i+1}. $\boldsymbol{\Lambda}$ is unique for a given $\boldsymbol{D}$.

There are many algorithms for computing the Smith normal form efficiently. Among all algorithms, Storijohann's algorithm, which is by using the modular techniques to compute the Smith normal form and transformation matrices, gives the best known complexity analysis for the integer matrices [22].

Proposition 4: Let $\boldsymbol{D} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{V}$ be the Smith normal form. Then $\boldsymbol{U}$ induces a group isomorphism $\tilde{\Phi} : \mathcal{G}(\boldsymbol{\Lambda}) \rightarrow \mathcal{G}(\boldsymbol{D})$ such that $\tilde{\Phi} : [\boldsymbol{a}]_\Lambda \in \mathcal{G}(\boldsymbol{\Lambda}) \mapsto [\boldsymbol{U}\boldsymbol{a}]_D \in \mathcal{G}(\boldsymbol{D})$.

Proof: By Proposition 3, we may assume $\boldsymbol{D} = \boldsymbol{U}\boldsymbol{\Lambda}$. Since $\boldsymbol{U}$ is an unimodular integer matrix, there is a group isomorphism $\Phi : \mathbb{Z}^M \rightarrow \mathbb{Z}^M$ such that $\Phi : \boldsymbol{a} \in \mathbb{Z}^M \mapsto \boldsymbol{U}\boldsymbol{a} \in \mathbb{Z}^M$. Let $\phi_{\boldsymbol{A}}$ be a group homomorphism from $\mathbb{Z}^M$ onto $\mathcal{G}(\boldsymbol{A})$ such that $\phi_{\boldsymbol{A}} : \boldsymbol{a} \in \mathbb{Z}^M \mapsto [\boldsymbol{a}]_A \in \mathcal{G}(\boldsymbol{A})$. Since the kernel of $\phi_{\boldsymbol{\Lambda}}$ is isomorphic to the kernel of $\phi_{\boldsymbol{D}}$ under the map Φ, it induces a canonical isomorphism $\tilde{\Phi} : \mathcal{G}(\boldsymbol{\Lambda}) \rightarrow \mathcal{G}(\boldsymbol{D})$ such that $\tilde{\Phi} : [\boldsymbol{a}]_\Lambda \in \mathcal{G}(\boldsymbol{\Lambda}) \mapsto [\boldsymbol{U}\boldsymbol{a}]_D \in \mathcal{G}(\boldsymbol{D})$ making the following diagram commutative:

$$\begin{array}{ccc} \mathbb{Z}^M & \xrightarrow{\Phi} & \mathbb{Z}^M \\ \downarrow \phi_{\boldsymbol{\Lambda}} & & \downarrow \phi_{\boldsymbol{D}} \\ \mathcal{G}(\boldsymbol{\Lambda}) & \xrightarrow{\tilde{\Phi}} & \mathcal{G}(\boldsymbol{D}) \end{array}$$

■

Remark 3: Since $\boldsymbol{\Lambda}$ is a diagonal matrix, then an obvious set of representatives of $\mathcal{G}(\boldsymbol{\Lambda})$ can be written as $\{[(a_1, a_2, ..., a_M)^T]_\Lambda \mid 0 \leq a_i < \lambda_i, a_i \in \mathbb{Z}\}$ where λ_i is the ith diagonal element of $\boldsymbol{\Lambda}$.

Example 2: In this example, we would like to find a set of representatives of $\mathcal{G}(\boldsymbol{D})$ for $\boldsymbol{D} = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$. By the Smith normal form decomposition, the sampling matrix $\boldsymbol{D} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{V}$ where $\boldsymbol{U} = \begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix}$, $\boldsymbol{\Lambda} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$, and $\boldsymbol{V} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$. It is easy to find that $\{[(0\ 0)^T]_\Lambda, [(0\ 1)^T]_\Lambda, [(0\ 2)^T]_\Lambda\}$ is a set of representatives of $\mathcal{G}(\boldsymbol{\Lambda})$. Then, by Proposition 4, a set of representatives of $\mathcal{G}(\boldsymbol{D})$ is $\{[(0\ 0)^T]_D, [(1\ 0)^T]_D, [(2\ 0)^T]_D\}$.

VI. Existence of PR Synthesis Polyphase Matrices

In this section, a necessary condition for the posed reconstructability problem with some sampling matrices and some synthesis polyphase matrices is discussed.

Definition 2: Let $H_0(\boldsymbol{z}), H_1(\boldsymbol{z}), ..., H_{N-1}(\boldsymbol{z})$ be polynomial analysis filters. Then the system is *polynomial (resp.: Laurent polynomial) perfectly reconstructable* if there exists some sampling matrices and some polynomial (resp.: Laurent polynomial) synthesis polyphase matrices, which satisfies the PR condition.

Proposition 5: Polynomial analysis filters $H_0(\boldsymbol{z})$, $H_1(\boldsymbol{z})$,...,$H_{N-1}(\boldsymbol{z})$ have no common zeros if and only if the system is polynomial perfectly reconstructable.

Proof: Suppose polynomial analysis filters $H_0(\boldsymbol{z}), H_1(\boldsymbol{z}), ..., H_{N-1}(\boldsymbol{z})$ have no common zero. To show the system is polynomial perfectly reconstructable, it is enough to show that there exists some polynomial synthesis polyphase matrices that have PR when the sampling matrix $\boldsymbol{D} = \boldsymbol{I}$. Since $H_0(\boldsymbol{z})$, $H_1(\boldsymbol{z})$,...,$H_{N-1}(\boldsymbol{z})$ have no common zeros and by the Weak Nullstellensatz Theorem [23], the submodule $\langle H_0(\boldsymbol{z}), H_1(\boldsymbol{z}), ..., H_{N-1}(\boldsymbol{z})\rangle$ of $\mathbb{C}[z]$ contains an unit. Therefore there exists polynomials $G_i(\boldsymbol{z})$ such that $\sum_{i=0}^{N-1} G_i(\boldsymbol{z})H_i(\boldsymbol{z}) = 1$. Therefore, the system is polynomial perfectly reconstructable. [1] Conversely, we want to show that if polynomial analysis filters $H_0(\boldsymbol{z}), H_1(\boldsymbol{z}), ..., H_{N-1}(\boldsymbol{z})$ have a common zero, then the system does not provide PR for any sampling matrices and any polynomial synthesis polyphase matrices. Suppose there exists $\boldsymbol{D}$ and $\boldsymbol{G}(\boldsymbol{z})$ satisfying the PR condition. By polyphase decomposition, we have $\boldsymbol{G}(\boldsymbol{z})\boldsymbol{H}(\boldsymbol{z}) = \boldsymbol{I}$. It is always true that

$$\boldsymbol{G}(\boldsymbol{z}^{\boldsymbol{D}})\boldsymbol{H}(\boldsymbol{z}^{\boldsymbol{D}}) = \boldsymbol{I}. \tag{4}$$

Let $\boldsymbol{r}(\boldsymbol{z})$ be the first row of $\boldsymbol{G}(\boldsymbol{z}^{\boldsymbol{D}})$ and $\boldsymbol{c}_j(\boldsymbol{z}^{\boldsymbol{D}})$ be the jth column of $\boldsymbol{H}(\boldsymbol{z}^{\boldsymbol{D}})$. Let

$$\boldsymbol{v}(\boldsymbol{z}) = \sum_{[\boldsymbol{l}_j]_D \in \mathcal{G}(\boldsymbol{D})} \boldsymbol{z}^{-\boldsymbol{l}_j} \boldsymbol{c}_j(\boldsymbol{z}^{\boldsymbol{D}}). \tag{5}$$

Without loss of generality, we may assume $\boldsymbol{l}_0 = \boldsymbol{0}$. Then the dot product is $\boldsymbol{r}(\boldsymbol{z}) \cdot \boldsymbol{v}(\boldsymbol{z}) = 1$ for all $\boldsymbol{z}$ by (4). But, by (3), we have

$$\boldsymbol{v}(\boldsymbol{z}) = \sum_{[\boldsymbol{l}_j]_D \in \mathcal{G}(\boldsymbol{D})} \boldsymbol{z}^{-\boldsymbol{l}_j} \boldsymbol{c}_j(\boldsymbol{z}^{\boldsymbol{D}}) = \begin{pmatrix} H_0(\boldsymbol{z}) \\ \vdots \\ H_{N-1}(\boldsymbol{z}) \end{pmatrix}. \tag{6}$$

Let $\boldsymbol{z}_0$ be a common zero of $\{H_i(\boldsymbol{z})\}$. Then $\boldsymbol{r}(\boldsymbol{z}_0) \cdot \boldsymbol{v}(\boldsymbol{z}_0) = 0 \neq 1$ which leads to a contradiction. ■

Definition 3: [17] A point in $\mathbb{C}^M$ is said to be *weak-zero* if at least one of its coordinates is zero.

Proposition 6: Let $H_0(\boldsymbol{z}), H_1(\boldsymbol{z}), ..., H_{N-1}(\boldsymbol{z})$ be polynomial analysis filters. The common solution set of $\{H_0(\boldsymbol{z}) = 0, H_1(\boldsymbol{z}) = 0, ..., H_{N-1}(\boldsymbol{z}) = 0\}$ contains only weak-zeros if and only if the system is Laurent polynomial perfectly reconstructable.

Proof: To show the system is Laurent polynomial perfectly reconstructable, we only need to show that there exists some Laurent polynomial synthesis polyphase matrices such that the system is PR when the sampling matrix $\boldsymbol{D} = \boldsymbol{I}$. Suppose $H_i(\boldsymbol{z})$ have no common solution, then by Proposition 5 the system has PR for some sampling matrices and some polynomial synthesis polyphase matrices. Now consider $H_i(\boldsymbol{z})$ have common solutions. Let $f(\boldsymbol{z}) = z_1 z_2 ... z_M$. Then $f(s) = 0$ for any s in the solution set of $\{H_i(\boldsymbol{z})\}$ because s is a weak-zero. By Hilbert's Nullstellensatz Theorem [24], there exists a positive integer m such that $f(\boldsymbol{z})^m \in \langle H_0(\boldsymbol{z}), H_1(\boldsymbol{z}), ..., H_{N-1}(\boldsymbol{z})\rangle$. Therefore there exists polynomials $G_i(\boldsymbol{z})$ such that $\sum_{i=0}^{N-1}(z_1 z_2 ... z_M)^{-m} G_i(\boldsymbol{z})H_i(\boldsymbol{z}) = 1$. This shows that the system is Laurent polynomial perfectly reconstructable. The other direction of the proof is similar to the proof in Proposition 5. ■

Example 3: Let $H_0(\boldsymbol{z}) = (1 + z_1)(1 + z_2)$,$H_1(\boldsymbol{z}) = (1 - z_1)(1 - z_1 z_2)$, $H_2(\boldsymbol{z}) = (1 - z_1)(z_1 - z_2)$, and $H_3(\boldsymbol{z}) =$

[1] Alternative proof of "only if" is given by Kalker and Shah. In the formulation of [19] the existence of a common zero is equivalent to the existence of $\boldsymbol{z}^*$ such that $\boldsymbol{H}(\boldsymbol{z}^*)\boldsymbol{1} = \boldsymbol{0}$, where $\boldsymbol{1}$ is the all-one vector. It follows that we cannot have $\boldsymbol{G}(\boldsymbol{z}^*)\boldsymbol{H}(\boldsymbol{z}^*) = \boldsymbol{I}$.

$(1-z_2)(1-z_1z_2)$ be the analysis filters. Determine whether the system is perfectly reconstructable or not?

Since the common zeros of $H_i(\boldsymbol{z})$ is $\{(-1,-1)\}$, this implies that it is not perfectly reconstructable. However, if we change $H_0(\boldsymbol{z}) = (1+pz_1)(1+qz_2)$ for some $p, q \neq \pm 1$, then the system will become perfectly reconstructable.

VII. Search for Maximum Density Sampling Matrix

In this section, we want to solve two problems. The first problem is how can we search for a maximum density sampling matrix. The second problem is what is the (worst case) iteration for the main algorithm mentioned in Section II.

The following theorem will tell us what is the size of $C_M(P)$.

Theorem 3: [21] Let $|C_M(P)|$ denote the size of $C_M(P)$, then

$$|C_1(P)| = 1,$$
$$|C_M(P)| = \sum_{q|P} q|C_{M-1}(q)|, \quad M \geq 2.$$

When $M = 2$, then we have $|C_2(P)| = \sum_{q|P} q$. Robin [25] proved that $|C_2(P)| = O(P \log\log P)$ where $O(\cdot)$ denotes the big O notation.

Example 4: If $M = 2$ and $P = 4$, then $|C_2(4)| = \sum_{q|4} q = 1+2+4 = 7$, which is same as in Example 1.

Remark 4: Suppose now that $P > N$. Since the rank of the matrix $\boldsymbol{H}(\boldsymbol{z})$ is N, it is less than the number of columns P. Therefore it is impossible to have a left inverse matrix if $P > N$. This provides us an upper bound of the search method.

According to Theorem 3 and Remark 4, the search for a sampling matrix such that the system is PR is a finite process. First we can set $P := N$. Then now we can check the invertibility of $\boldsymbol{H}(\boldsymbol{z})$ corresponding to each $\boldsymbol{D} \in C_M(P)$. If none of them is invertible, then we reduce the value P by one and continue the process again, else we output $\boldsymbol{D}$. This procedure ensures us that it would return a maximum density sampling matrix.

By Theorem 3, the maximum number of iterations for the main algorithm is $\sum_{m=1}^{N}\sum_{q|m} q \cdot |C_{M-1}(q)|$ for $M \geq 2$ and $N(N+1)/2$ for $M = 1$. In particular, when $M = 2$, then the worst case is $\sum_{m=1}^{N}\sum_{q|m} q = \frac{\pi^2}{12}N^2 + O(N \ln N)$ by Handy and Weight [26].

Example 5: Let $H_0(\boldsymbol{z}) = (1+z_1)(1+z_2), H_1(\boldsymbol{z}) = (1-z_1)(1-z_1z_2), H_2(\boldsymbol{z}) = (1-z_1)(z_1-z_2), H_3(\boldsymbol{z}) = (1-z_2)(1-z_1z_2), H_4(\boldsymbol{z}) = (1-z_2)(z_1-z_2), H_5(\boldsymbol{z}) = (1-z_1)(1-z_2)$ be the analysis filters. By our main algorithm, one can show that the solution set of analysis filters is an empty set, which implies the system is polynomial perfectly reconstructable. One can obtain a sampling matrix $\boldsymbol{D} = \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix}$ with maximum sampling rate and synthesis polyphase matrix $\boldsymbol{G}(\boldsymbol{z}) =$

$$\begin{pmatrix} -\frac{z_1}{8}+\frac{3}{8} & -\frac{z_1z_2}{24}-\frac{z_1}{12}+\frac{z_2}{8}+\frac{1}{4} & -\frac{z_1z_2}{12}-\frac{z_1}{24}+\frac{z_2}{4}+\frac{1}{8} \\ \frac{1}{4} & \frac{z_2}{12}+\frac{1}{2} & \frac{z_2}{6}+\frac{3}{4} \\ 0 & \frac{z_2}{3} & \frac{2z_2}{3} \\ \frac{1}{4} & \frac{z_1}{3}+\frac{z_2}{12}-\frac{1}{2} & \frac{2z_1}{3}+\frac{z_2}{6}-\frac{5}{4} \\ 0 & -\frac{2z_1}{3} & -\frac{4z_1}{3}+1 \\ \frac{z_1}{8}+\frac{1}{8} & \frac{z_1z_2}{24}-\frac{z_1}{4}-\frac{7z_2}{24}-\frac{1}{4} & \frac{z_1z_2}{12}-\frac{5z_1}{8}-\frac{7z_2}{12}+\frac{3}{8} \end{pmatrix}^T$$

which satisfies the PR condition.

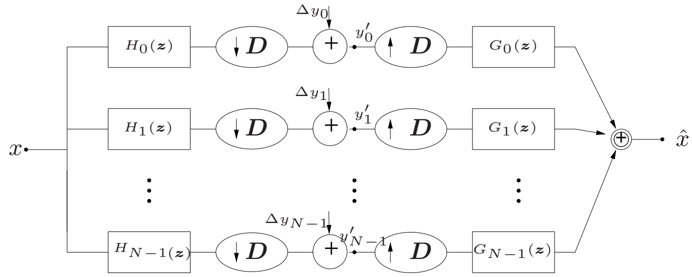


Fig. 4. An N-channel filter bank with possible additive noise.

VIII. Optimizations

In this section, we assume the analysis filters and sampling matrix are given. Our objective is to find an optimal PR synthesis polyphase matrix for which the system can minimize the effect of noise in the reconstructed signal shown in Fig. 4.

A. Inner Products

In this subsection, we provide one criterion (i.e., minimizing $\|\boldsymbol{G}(\boldsymbol{z})\|_S$) on the PR synthesis matrices to reduce the noise level. First, we briefly define the notion of inner products over infinite sequences and Laurent polynomial matrices.

We define an inner product $\langle \cdot, \cdot \rangle$ on a column of J-multidimensional infinite sequences over $\mathbb{C}$ such that

$$\langle \boldsymbol{a}, \boldsymbol{a}' \rangle = \sum_{i=0}^{K} \sum_{\boldsymbol{n} \in \mathbb{Z}^J} a_i[\boldsymbol{n}] a_i'^*[\boldsymbol{n}]. \tag{7}$$

where $\boldsymbol{a} = (a_0, a_1, ..., a_K), \boldsymbol{a}' = (a'_0, a'_1, ..., a'_K)$ are J-multidimensional infinite sequences. In the later context, we will use $\|\cdot\|_1$ and $\|\cdot\|_2$ to denote the matrix one norm and two norm, respectively. To avoid the confusion of the notations, we denote $\|\boldsymbol{a}\|_E := \sqrt{\sum_{i=0}^{K}\sum_{\boldsymbol{n}\in\mathbb{Z}^J} a_i[\boldsymbol{n}] a_i{}^*[\boldsymbol{n}]}$ to be the Euclidean Norm. We define an inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{C}[z_1, ..., z_J]^{K \times L}$ over $\mathbb{C}$ such that

$$\langle \boldsymbol{A}(\boldsymbol{z}), \boldsymbol{A}'(\boldsymbol{z}) \rangle = \int_{[0,1)^J} \operatorname{tr}\Big(\boldsymbol{A}'(e^{2\pi j\boldsymbol{\theta}})^H \boldsymbol{A}(e^{2\pi j\boldsymbol{\theta}})\Big)\, d\boldsymbol{\theta} \tag{8}$$

where $(\cdot)^H$ and $\operatorname{tr}(\cdot)$ denote the conjugate transpose and trace of a matrix, respectively. To overload the notation, we denote $\|\cdot\|_E := \sqrt{\langle \cdot, \cdot \rangle}$ for all inner products defined above.

Remark 5: By Parseval's theorem, we have $\|\boldsymbol{X}(\boldsymbol{z})\|_E^2 = \|x\|_E^2, \|\boldsymbol{G}(\boldsymbol{z})\|_E^2 = \|\boldsymbol{g}\|_E^2$, and $\|\boldsymbol{Y}(\boldsymbol{z})\|_E^2 = \|\boldsymbol{y}\|_E^2$.

Definition 4: We define $\|\cdot\|_S$ to be the essential supremum of the two norm on the unit sphere. i.e.,

$$\|\boldsymbol{R}(\boldsymbol{z})\|_S := \operatorname{ess\,sup}_{\boldsymbol{\theta}\in[0,1)^M} \|\boldsymbol{R}(e^{2\pi j\boldsymbol{\theta}})\|_2. \tag{9}$$

Proposition 7: Let $\hat{\boldsymbol{X}}(\boldsymbol{z}), \boldsymbol{G}(\boldsymbol{z})$, and $\boldsymbol{Y}(\boldsymbol{z})$ be the polyphase matrices of the output signal, the synthesis filters, and the sub-band signals, respectively, such that $\hat{\boldsymbol{X}}(\boldsymbol{z}) = \boldsymbol{G}(\boldsymbol{z})\boldsymbol{Y}(\boldsymbol{z})$. Then $\|\hat{\boldsymbol{X}}(\boldsymbol{z})\|_E \leq \|\boldsymbol{G}(\boldsymbol{z})\|_S \|\boldsymbol{Y}(\boldsymbol{z})\|_E$.

Proof: By Remark 5, we have

$$\begin{aligned} &\|\hat{\boldsymbol{X}}(\boldsymbol{z})\|_E^2 \\ =&\int_{[0,1)^M} \boldsymbol{Y}(e^{2\pi j\boldsymbol{\theta}})^H \boldsymbol{G}(e^{2\pi j\boldsymbol{\theta}})^H \boldsymbol{G}(e^{2\pi j\boldsymbol{\theta}}) \boldsymbol{Y}(e^{2\pi j\boldsymbol{\theta}})\, d\boldsymbol{\theta} \\ \leq& \|\boldsymbol{G}(\boldsymbol{z})\|_S^2 \|\boldsymbol{Y}(\boldsymbol{z})\|_E^2 \end{aligned} \tag{10}$$

■

The following corollary suggests us that we can reduce the noise level, if we minimize $\|\cdot\|_S$ with respect to $\boldsymbol{G}(\boldsymbol{z})$.

Corollary 1: Let $\boldsymbol{G}(\boldsymbol{z})$ be a PR synthesis polyphase matrix for a given analysis polyphase matrix $\boldsymbol{H}(\boldsymbol{z})$. Let $\Delta \boldsymbol{y}$ be an perturbation of the sub-band signals. Then

$$\|\hat{x} - x\|_E \leq \|\boldsymbol{G}(\boldsymbol{z})\|_S \|\Delta \boldsymbol{y}\|_E. \tag{11}$$

B. Characterization and Finite Support

In this subsection, we will show that even though the para-pseudoinverse $\boldsymbol{H}^\dagger(\boldsymbol{z})$ is the optimal solution among the inverses with respect to the Euclidean norm $\|\cdot\|_E$ and the essential supremum two norm on the unit sphere $\|\cdot\|_S$, it is not a good choice in practice.

The following theorem characterizes the set of all inverses.

Theorem 4: [17] Suppose $\boldsymbol{H}(\boldsymbol{z})$ is an $N \times P$ polynomial matrix and $\tilde{\boldsymbol{G}}(\boldsymbol{z})$ is a $P \times N$ polynomial (resp.: Laurent polynomial) matrix such that $\tilde{\boldsymbol{G}}(\boldsymbol{z})\boldsymbol{H}(\boldsymbol{z}) = \boldsymbol{I}$. Then $\boldsymbol{G}(\boldsymbol{z})$ is an polynomial (resp.: Laurent polynomial) inverse matrix of $\boldsymbol{H}(\boldsymbol{z})$ if and only if $\boldsymbol{G}(\boldsymbol{z})$ can be written as

$$\boldsymbol{G}(z) = \tilde{\boldsymbol{G}}(z) + \boldsymbol{A}(z)(\boldsymbol{I} - \boldsymbol{H}(z)\tilde{\boldsymbol{G}}(z)) \tag{12}$$

where $\boldsymbol{A}(\boldsymbol{z})$ is an arbitrary $P \times N$ polynomial (resp.: Laurent polynomial) matrix. (Note: We can extend this theorem to the matrices over any ring.)

Definition 5: The para-pseudoinverse of $\boldsymbol{H}(\boldsymbol{z})$ is defined as

$$\boldsymbol{H}^\dagger(z) = \Big(\boldsymbol{H}^H(\boldsymbol{z}^{-1})\boldsymbol{H}(\boldsymbol{z})\Big)^{-1}\boldsymbol{H}^H(\boldsymbol{z}^{-1}). \tag{13}$$

Proposition 8: Suppose the para-pseudoinverse $\boldsymbol{H}^\dagger(\boldsymbol{z})$ is well defined. Then $\boldsymbol{H}^\dagger(\boldsymbol{z})$ is a PR synthesis polyphase matrix and $\boldsymbol{H}^\dagger(\boldsymbol{z})$ is the minimum with respect to $\|\cdot\|_S$ among all the PR synthesis polyphase matrices.

Proof: For every $\boldsymbol{\theta}$ is on the unit sphere, an optimal solution is $\boldsymbol{H}^\dagger(e^{2\pi j\boldsymbol{\theta}})$ among all inverses with respect to $\|\cdot\|_2$. Therefore, $\boldsymbol{H}^\dagger(\boldsymbol{z})$ is the minimum with respect to $\|\cdot\|_S$ among all the PR synthesis polyphase matrices. ■

Proposition 9: Suppose the para-pseudoinverse $\boldsymbol{H}^\dagger(\boldsymbol{z})$ is well defined. Then $\boldsymbol{H}^\dagger(\boldsymbol{z})$ is the minimum with respect to $\|\cdot\|_E$ among all the PR synthesis polyphase matrices.

Proof: Using (12), we have $\|\boldsymbol{G}(\boldsymbol{z})\|_E = \|\boldsymbol{H}^\dagger(\boldsymbol{z})\|_E + \|\boldsymbol{A}(\boldsymbol{z})(\boldsymbol{I} - \boldsymbol{H}(\boldsymbol{z})\boldsymbol{H}^\dagger(\boldsymbol{z}))\|_E$ because

$$\begin{aligned} &\langle \boldsymbol{H}^\dagger(\boldsymbol{z}), \boldsymbol{I} - \boldsymbol{H}(\boldsymbol{z})\boldsymbol{H}^\dagger(\boldsymbol{z})\rangle \\ =&\int_{[0,1)^M} \mathrm{tr}((\boldsymbol{I} - \boldsymbol{H}(e^{2\pi j\boldsymbol{\theta}})\boldsymbol{H}^\dagger(e^{2\pi j\boldsymbol{\theta}}))^H \boldsymbol{H}^\dagger(e^{2\pi j\boldsymbol{\theta}}))\, d\boldsymbol{\theta} \\ =&\int_{[0,1)^M} \mathrm{tr}(\boldsymbol{H}^\dagger(e^{2\pi j\boldsymbol{\theta}})(\boldsymbol{I} - \boldsymbol{H}^\dagger(e^{2\pi j\boldsymbol{\theta}})^H \boldsymbol{H}(e^{2\pi j\boldsymbol{\theta}})^H))\, d\boldsymbol{\theta} \\ =&\int_{[0,1)^M} \mathrm{tr}(\boldsymbol{H}^\dagger(e^{2\pi j\boldsymbol{\theta}}) - (\boldsymbol{H}^H(e^{-2\pi j\boldsymbol{\theta}})\boldsymbol{H}(e^{2\pi j\boldsymbol{\theta}}))^{-1}\boldsymbol{H}(e^{2\pi j\boldsymbol{\theta}})^H)\, d\boldsymbol{\theta} = 0. \end{aligned} \tag{14}$$

Therefore $\boldsymbol{H}^\dagger(\boldsymbol{z})$ is the minimum with respect to $\|\cdot\|_E$ among all the PR synthesis polyphase matrices. ■

Remark 6: Let $\boldsymbol{H}(\boldsymbol{z})$ be an analysis polyphase matrix with full rank. If $\det\big(\boldsymbol{H}^H(\boldsymbol{z}^{-1})\boldsymbol{H}(\boldsymbol{z})\big)$ is not a monomial, then $\boldsymbol{H}^\dagger(\boldsymbol{z})$ is not Laurent polynomial matrix (i.e., the PR synthesis filters $\boldsymbol{h}^\dagger$ do not have finite support).

Though $\boldsymbol{H}^\dagger(\boldsymbol{z})$ is the minimum with respect to $\|\cdot\|_S$ or $\|\cdot\|_E$ among all the PR synthesis polyphase matrices, $\boldsymbol{H}^\dagger(\boldsymbol{z})$ may not be a practical PR synthesis polyphase matrix to use. The reason is that the probability of $\det\big(\boldsymbol{H}^H(\boldsymbol{z}^{-1})\boldsymbol{H}(\boldsymbol{z})\big)$ being a monomial is zero. Thus, $\boldsymbol{H}^\dagger(\boldsymbol{z})$ usually is not a Laurent polynomial matrix.

C. Optimization of Synthesis Laurent Polynomial Polyphase Matrices

Although it would be ideal to find an optimal solution by minimizing $\|\cdot\|_S$ with respect to $\boldsymbol{G}(\boldsymbol{z})$ for a given support of $\boldsymbol{A}(\boldsymbol{z})$, it is impractical to obtain such a solution because we need to evaluate every point on the unit sphere by the definition of $\|\cdot\|_S$. However, minimizing with respect to $\frac{1}{\sqrt{P}}\|\boldsymbol{G}(\boldsymbol{z})\|_E$ or $\sqrt{N}\max_{i=0,\dots,N-1}\|g_i\|_1$ would be feasible in practice. Moreover, by the following propositions, we show that the value of $\|\boldsymbol{G}(\boldsymbol{z})\|_S$ is between the two values.

Proposition 10: [27], [28] Let $\boldsymbol{G}(\boldsymbol{z})$ be a synthesis polyphase matrix. Then

$$\frac{1}{\sqrt{P}}\|\boldsymbol{G}(\boldsymbol{z})\|_E \leq \|\boldsymbol{G}(\boldsymbol{z})\|_S. \tag{15}$$

By (12), we can express $g_i[\boldsymbol{n}]$ in terms of linear combination of $a_{i,j}[\boldsymbol{k}]$ and $\|\boldsymbol{g}\|_E$ depends quadratically on $a_{i,j}[\boldsymbol{k}]$. Therefore, to obtain the minimum value of $\|\boldsymbol{g}\|_E$, one can do the partial derivative of $\|\boldsymbol{g}\|_E$ with respect to $a_{i,j}[\boldsymbol{k}]$ and set them all equal to zero, and then solve the linear equations. Here, we provide an algorithm to find an optimal inverse with respect to $\|\cdot\|_E$ shown in Algorithm 3.

Proposition 11: Suppose additive white Gaussian independent noise $\boldsymbol{\epsilon}$ is imposed on the sub-band signals $\boldsymbol{y}$ with zero mean and power density σ^2. Suppose $\mathcal{Q}$ is a given set of the inverses of $\boldsymbol{H}(\boldsymbol{z})$. Then the minimum mean square error among all the left inverses in $\mathcal{Q}$ is

$$\mathrm{MMSE}(\mathcal{Q}) = \frac{\sigma^2}{P}\min_{\boldsymbol{G}(\boldsymbol{z})\in\mathcal{Q}}\|\boldsymbol{G}(\boldsymbol{z})\|_E^2. \tag{16}$$

Proof: Let $\boldsymbol{\eta} = \boldsymbol{G}(\boldsymbol{z})\boldsymbol{\epsilon}$ be the reconstruction error. Since the noise is Gaussian independent, $\mathbb{E}[\epsilon_i\epsilon_i^H] = \sigma^2$ and $\mathbb{E}[\epsilon_i\epsilon_j^H] = 0$ for $i \neq j$. Then the covariance of $\boldsymbol{\eta}$ is $R_{\boldsymbol{\eta}\boldsymbol{\eta}} = \mathbb{E}[\boldsymbol{\eta}\boldsymbol{\eta}^H] = \sigma^2\mathbb{E}[\boldsymbol{G}(\boldsymbol{z})\boldsymbol{G}(\boldsymbol{z})^H]$. The minimum mean square error among all left inverses in $\mathcal{Q}$ is

$$\begin{aligned} \mathrm{MMSE}(\mathcal{Q}) &= \min_{\boldsymbol{G}(\boldsymbol{z})\in\mathcal{Q}}\frac{1}{P}\sum_{i=1}^{P}\mathbb{E}[\eta_i^2] \\ &= \frac{\sigma^2}{P}\min_{\boldsymbol{G}(\boldsymbol{z})\in\mathcal{Q}}\int_{[0,1)^M}\mathrm{tr}\Big(\boldsymbol{G}(e^{2\pi j\boldsymbol{\theta}})\boldsymbol{G}(e^{2\pi j\boldsymbol{\theta}})^H\Big)d\boldsymbol{\theta} \\ &= \frac{\sigma^2}{P}\min_{\boldsymbol{G}(\boldsymbol{z})\in\mathcal{Q}}\|\boldsymbol{G}(\boldsymbol{z})\|_E^2. \end{aligned} \tag{17}$$

■

This implies that Algorithm 3 gives us the PR Laurent polynomial synthesis polyphase matrix with the minimum mean square error for a given support of $\boldsymbol{A}(\boldsymbol{z})$. However, if the additive noise is not Gaussian independent, then Algorithm 3 does not guarantee the solution having the best performance in terms of mean square error. Therefore, in the following part, we provide another algorithm, which readers can compare with Algorithm 3 and determine which perform better under the particular noise.

Proposition 12: [29] Let $\boldsymbol{G}(\boldsymbol{z})$ be a synthesis polyphase matrix. Then

$$\|\boldsymbol{G}(\boldsymbol{z})\|_S \leq \sqrt{N} \max_{i=0,...,N-1} \|g_i\|_1 \tag{18}$$

where g_i corresponds to the ith synthesis filter.

Proof: Followed by [29, p.314] and by the definition of one norm $\|\cdot\|_1$. ∎

By Proposition 12 and Corollary 1, we know $\|\hat{x} - x\|_E \leq \sqrt{N} \max_{i=0,...,N-1} \|g_i\|_1 \|\Delta \boldsymbol{y}\|_E$. To reduce the noise level, we can minimize $\max_{i=0,...,N-1} \|g_i\|_1$ (i.e., $\max_{i=0,...,N-1} \sum_{\boldsymbol{m}\in\mathcal{S}} |g_i[\boldsymbol{m}]|$ where $g_i[\boldsymbol{m}]$ is in terms of the linear combination of $a_{i,j}[\boldsymbol{k}]$ by (12). Since we assume $\boldsymbol{A}(\boldsymbol{z})$ has a finite support, this implies the support $\mathcal{S}$ of $\boldsymbol{g}$ is also a finite set). The optimization problem can be expressed as

$$\begin{aligned} &\text{minimize} \quad w \\ &\text{subject to} \quad w - \sum_{\boldsymbol{m}\in\mathcal{S}} |g_i[\boldsymbol{m}]| \geq 0 \text{ for } i = 0, ..., N-1. \end{aligned}$$

This problem can be converted into the following form:

$$\begin{aligned} \text{minimize} \quad & w \\ \text{subject to} \quad & w - \sum_{\boldsymbol{m}\in\mathcal{S}} e_i[\boldsymbol{m}] \geq 0 \text{ for } i = 0, ..., N-1, \\ & e_i[\boldsymbol{m}] + g_i[\boldsymbol{m}] \geq 0 \text{ for } i = 0, ..., N-1, \ \boldsymbol{m}\in\mathcal{S}, \\ & e_i[\boldsymbol{m}] - g_i[\boldsymbol{m}] \geq 0 \text{ for } i = 0, ..., N-1, \ \boldsymbol{m}\in\mathcal{S}, \end{aligned} \tag{19}$$

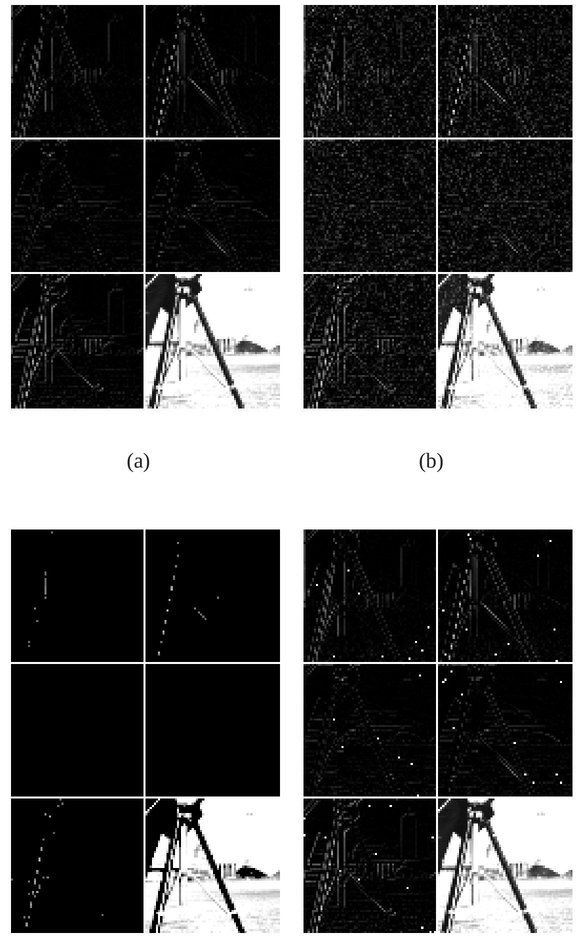
which is a linear optimization problem and can be solved by using the linear programming.

Algorithm 3 (resp.: Algorithm 4) Euclidean Norm Optimal Inverse (resp.: One-Norm Optimal Inverse)

Input: $N \times P$ matrix $\boldsymbol{H}(\boldsymbol{z})$ with M variables
Output: an optimal left inverse $P \times N$ matrix $\boldsymbol{G}(\boldsymbol{z})$ with respect to $\|\cdot\|_E$ (resp.: $\max_{i=0,...,N-1} \|g_i\|_1$) and a given finite support $\{Q_{i,j}\}$ of $\boldsymbol{A}(\boldsymbol{z})$ where $A_{i,j}(\boldsymbol{z}) = \sum_{\boldsymbol{k}\in Q_{i,j}} a_{i,j}[\boldsymbol{k}]\boldsymbol{z}^{\boldsymbol{k}}$

1: compute a particular inverse $\tilde{\boldsymbol{G}}(\boldsymbol{z})$ by Algorithm 2
2: parametrize $\boldsymbol{G}(\boldsymbol{z})$ with respect to $a_{i,j}[\boldsymbol{k}]$ using (12)
3: set all partial derivatives of $\|\boldsymbol{G}(\boldsymbol{z})\|_E$ with respect to $a_{i,j}[\boldsymbol{k}]$ to zero and solve the linear equations (resp.: 3: use the linear programming to solve the problem (19))
4: back substitute an optimal solution $\tilde{\boldsymbol{A}}(\boldsymbol{z})$ in $\boldsymbol{G}(\boldsymbol{z})$
5: output $\boldsymbol{G}(\boldsymbol{z})$

Example 6: We demonstrate a simulation of 6-channel filter bank with different types of additive noises. Our input signals is the cameraman image of size 256×256. Given analysis filters $H_0(\boldsymbol{z}) = (1-z_1)(1-z_1z_2)$, $H_1(\boldsymbol{z}) = (1-z_1)(z_1-z_2)$, $H_2(\boldsymbol{z}) = (1-z_2)(1-z_1z_2)$, $H_3(\boldsymbol{z}) = (1-z_2)(z_1-z_2)$, $H_4(\boldsymbol{z}) = (1-z_1^2z_2)(1-z_2^2z_1)$, and $H_5(\boldsymbol{z}) = (1+z_1)(1+z_2)$ and a sampling matrix $\boldsymbol{D} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and the support of $\boldsymbol{A}(\boldsymbol{z})$ given in (12) is $\{Q_{i,j}\}$ where $Q_{i,j} = \{\boldsymbol{0}\}$ for all i, j. Let Δy_i be additive noise on each row independently. Then we can represent the sub-band signal with noise y_i' shown in Fig. 4, in the polyphase domain, as $\boldsymbol{Y}'(\boldsymbol{z}) = \boldsymbol{H}(\boldsymbol{z})\boldsymbol{X}(\boldsymbol{z}) + \Delta\boldsymbol{Y}(\boldsymbol{z})$ where $\Delta\boldsymbol{Y}(\boldsymbol{z})$ be a polyphase matrix of Δy_i. We want to compare with the reconstruction performance in MSE between Algorithm 2 (i.e., a particular inverse), Algorithm 3 (i.e., $\|\cdot\|_E$ optimization), and Algorithm 4 (i.e., $\|\cdot\|_1$ optimization). We show the sub-band signals in Fig. 5(b) - 5(d) with different types of noises. The reconstruction images are shown in Fig. 6(b) - 6(i). We find out that for all different noises, Algorithm 2 performs the worst in MSE among all the algorithms. For Algorithm 3, it has the best performance in MSE with Gaussian noise and salt and pepper noise shown in Fig 7(a) and Fig 7(c). These results satisfy Proposition 11. For Algorithm



(a) (b) (c) (d)

Fig. 5. The origin sub-band signals and the different additive noise sub-band signals. (a) The clear sub-band signals (b) Additive Gaussian noise with zero mean and variance $\sigma = 0.01$ (c) Eliminating 85000 of insignificant coefficients (d) Additive salt and pepper noise with noise densities $d = 0.005$.

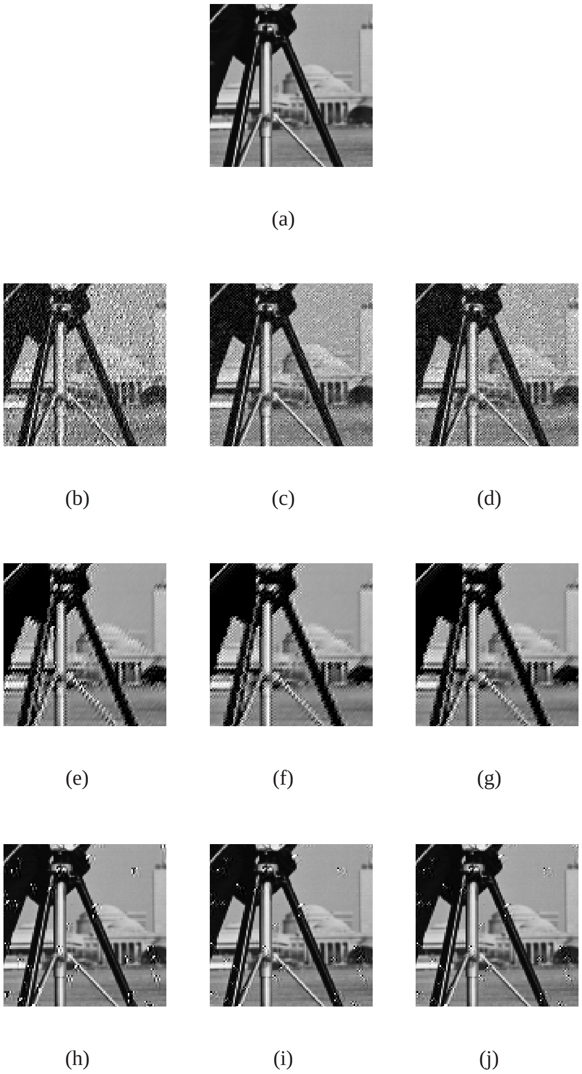


Fig. 6. The origin image and the reconstruction outputs with additive noise. (a) The original image. (b)-(d) Algorithm 2, Algorithm 3, and Algorithm 4 with Gaussian noise ($\sigma = 0.01$), MSE=0.0259, 0.0147, and 0.0157. (e)-(g) Algorithm 2, Algorithm 3, and Algorithm 4 with eliminating 85000 of insignificant coefficients, MSE=0.0082, 0.0063, and 0.0060. (h)-(j) Algorithm 2, Algorithm 3, and Algorithm 4 with salt and pepper noise (noise density 0.005), MSE=0.0145, 0.0058, and 0.0062.

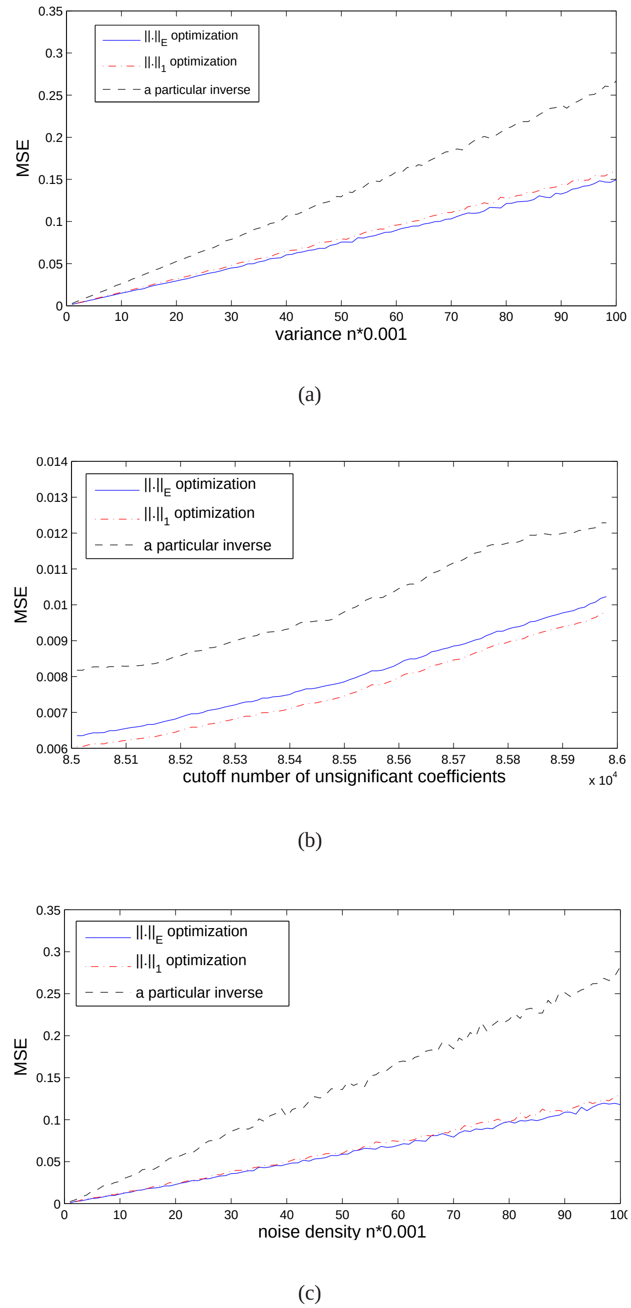


Fig. 7. MSE of the reconstruction errors. (a) Additive Gaussian noise with zero mean and different levels of variance (b) Eliminate the different numbers of insignificant coefficients (c) Additive salt and pepper noise with different noise densities.

4, it has the best performance in MSE if we eliminate different numbers of insignificant coefficients shown in Fig 7(b). This demonstrates that Algorithm 3 may not provide an optimal solution, if the noise is not white Gaussian independent.

IX. Conclusion

In this paper, we study the theory and algorithms of an optimal use of multidimensional signal reconstruction from multichannel acquisition using a filter bank setup. From Proposition 5 and Proposition 6, we address the necessary and sufficient condition on the analysis filters for a PR system with some sampling matrices and some synthesis polyphase matrices. Using the Hermite and Smith normal forms, we provide a search method to find a maximum density sampling matrix and compute the worst case iteration for our main algorithm. Once having a particular PR synthesis polyphase matrix, we can parametrize the set of all the PR synthesis polyphase matrices where an optimal solution can be obtained for given design criteria.

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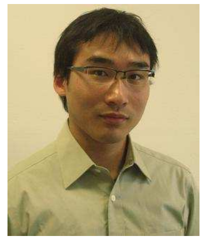

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