CONTOURLETS: A DIRECTIONAL MULTIRESOLUTION IMAGE REPRESENTATION

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ABSTRACT

We propose a new scheme, named *contourlet*, that provides a flexible multiresolution, local and directional image expansion. The contourlet transform is realized efficiently via a double iterated filter bank structure. Furthermore, it can be designed to satisfy the anisotropy scaling relation for curves, and thus offers a fast and structured curvelet-like decomposition. As a result, the contourlet transform provides a sparse representation for two-dimensional piecewise smooth signals resembling images. Finally, we show some numerical experiments demonstrating the potential of contourlets in several image processing tasks.

1. INTRODUCTION

Sparse signal expansion lies at the foundation of many signal processing tasks, including compression, filtering, and feature extraction. We are interested in the construction of sparse expansions for *two-dimensional signals which are smooth away from discontinuities across smooth curves*. Such signals resemble natural images where discontinuities are generated by *edges* – referred to points in the image with sharp contrast in the intensity, whereas edges are often gathered along smooth *contours* that are created by typically smooth boundaries of physical objects.

For one-dimensional piecewise smooth signals, wavelets provide the right tool. However, in 2-D the commonly used separable wavelets obtained by a tensor-product of 1-D wavelets are only good at capturing the discontinuities at edge points, but do not see the smoothness along contours. Thus, more powerful schemes are needed in higher dimensions.

Recently, Candès and Donoho [1] pioneered a new signal expansion, named *curvelet*, that offers a sparse expansion for 2-D piecewise smooth functions in $\mathbb{R}^2$ where the discontinuity curves are smooth. The original construction of the curvelet transform [1] was intended for functions defined in the *continuum* space $\mathbb{R}^2$. The development of *discrete* versions of the curvelet transform that can be applied to sampled images was a challenge, especially when critical sampling is desirable.

In this paper, first we identify the key features that lead to an improvement of curvelets over wavelets in representing 2-D piecewise smooth signals with smooth discontinuity curves. Based on this, we construct a new filter bank structure that can deal effectively with piecewise smooth images with smooth contours. The resulting image expansion is a frame composed of contour segments, and thus is named *contourlet*. Like wavelets, contourlets have a seamless translation between the continuous and the discrete worlds via a multiresolution analysis framework and iterated filter banks. Finally, we show some numerical experiments comparing wavelets and contourlets.

2. REPRESENTING 2-D PIECEWISE SMOOTH SIGNALS

Consider the wavelet transform of a 2-D piecewise functions with a smooth discontinuity curve (Fig. 1(a)). Due to the separable construction, 2-D wavelet basis functions have supports on dyadic squares. Consequently, wavelets are good at isolating discontinuity points as only wavelets whose supports overlap with the discontinuity curve generate significant coefficients. However, they are blind to the smoothness of this curve and it is easy to see that there are $O(2^j)$ significant wavelet coefficients at the scale 2^{-j}.

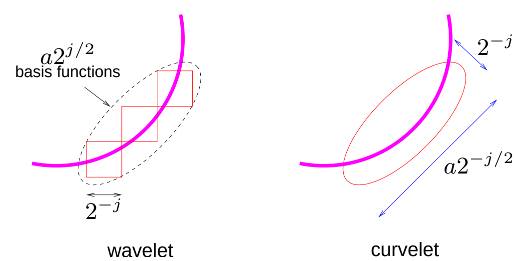

Fig. 1. Non-linear approximation of a 2-D piecewise smooth signals where the thick lines represent the discontinuity curves separating two smooth regions. Curvelet basis functions can be viewed as a local grouping of wavelet basis functions into linear structures so that they can capture the smooth discontinuity curve more efficiently.

How can we improve the performance of the wavelet scheme when the discontinuity curve is known to be smooth? Simply looking at the wavelet transform in Fig. 1(a) suggests that rather than treating each significant wavelet coefficient along the discontinuity curve independently, we can group the nearby coefficients as their locations are locally correlated. The curve scaling relation hints that we can group about $a2^{j/2}$ nearby wavelet basis functions at the scale 2^{-j} into one basis function with a linear structure so that its width is proportional to its length squared, as shown in Fig. 1(b). This grouping operation reduces the number of significant coefficients at the scale 2^{-j} from $O(2^j)$ to $O(2^{j/2})$. Consequently, such a new expansion is superior compared with the wavelet transform in approximating this type of 2-D piecewise smooth functions. This is the underlying reason for the success of the curvelet transform [1].

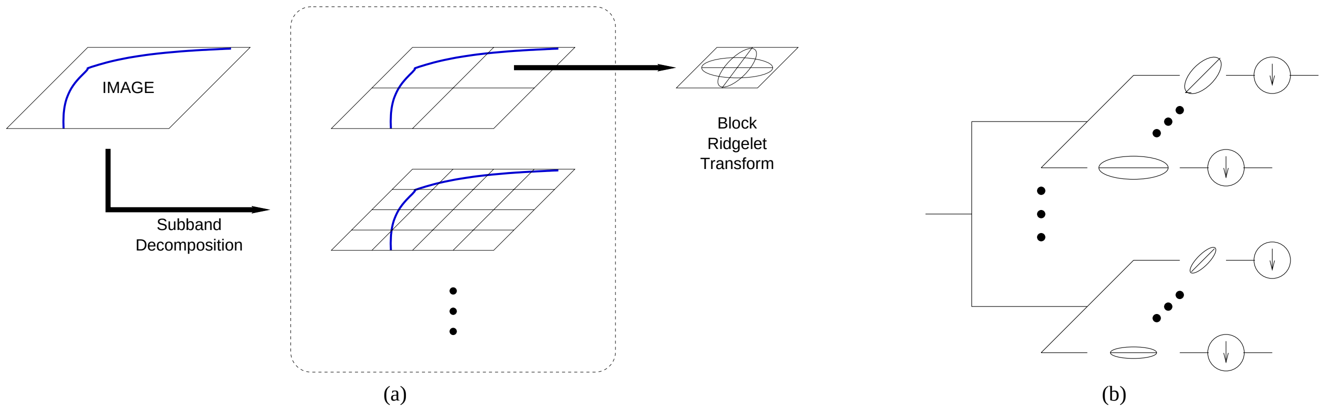

Fig. 2. Two approaches for dealing with piecewise smooth images. (a) *Curvelet construction:* block ridgelet transforms are applied to subband images. (b) *Contourlet construction:* image is decomposed by a double filter bank structure, where the first one captures the edge points and the second one link these edge points into contour segments.

To sum up, in order to provide sparse expansions for piecewise smooth images with smooth contours, in addition to localization and multiresolution features, new 2-D schemes should contain basis functions with elongated shapes with different aspect rations and oriented at variety of directions.

3. CONTOURLET CONSTRUCTION

The curvelet transform [1] achieves the aforementioned desiderata via filtering and then applying a block ridgelet transform on each bandpass image (see Fig. 2(a)). However, this construction poses several problems when one translates it into the discrete world. First, since it is a block-based transform, either the approximated images have blocking effects or we have to use overlapping windows and thus increase the redundancy. Secondly, the use of the ridgelet transform, which is defined on a polar coordinate, makes the implementation of the curvelet transform for discrete images on a rectangular coordinate to be very challenging. In [2, 3], different interpolation approaches were proposed to solve the polar versus rectangular coordinate transform problem, all requiring overcomplete systems. For example, the discrete implementation of the curvelet transform in [3] has a redundancy factor equal to $16J+1$, where J is the number of multiscale levels. This impose a serve limitation on curvelets in certain applications, such as compression.

The grouping of wavelet coefficients argument in the last section suggests that we can obtain a sparse image expansion by first applying a multiscale transform and then applying a local directional transform to gather the nearby basis functions at the same scale into linear structures. In essence, we use first a wavelet-like transform for *edge (points)* detection, and then a local directional transform for *contour segments* detection. This approach is similar to the popular Hough transform for line detection in computer vision.

With this insight, we construct a *double filter bank* structure (Fig. 2(b)) in which at first the Laplacian pyramid (LP) [4] is used to capture the point discontinuities, and followed by a directional filter bank (DFB) [5] to link point discontinuities into linear structures. The overall result is an image expansion with basis images as contour segments, and thus it is named the *contourlet transform*. The details of the proposed double filter bank, named *pyramidal directional filter bank* (PDFB), and its properties are given in [6].

Next, we sketch briefly the continuous-domain expansion generated by the contourlet construction and refer to [7] for details. Associated with the Laplacian pyramid is a multiscale decomposition of the $L^2(\mathbb{R}^2)$ space into a series of increasing resolution

$$L^2(\mathbb{R}^2) = V_{j_0} \oplus \left(\bigoplus_{j=j_0}^{-\infty} W_j \right), \tag{1}$$

with the usual definition of the subspaces V_{j_0} and W_j as in the wavelet multiresolution analysis [8] (Sec 7.1): V_{j_0} is an approximation subspace at the scale 2^{j_0}, whereas W_j contains the "added details" to the finer scale 2^{j-1}. In the LP, each subspace W_j is spanned by a frame $\{\mu_{j,n}(t)\}_{n\in\mathbb{Z}^2}$ that assimilates a uniform grid on $\mathbb{R}^2$ of intervals $2^{j-1} \times 2^{j-1}$.

For the directional filter bank, it can be shown that an l-level DFB generates a local directional basis for $l^2(\mathbb{Z}^2)$ that is composed of the impulse responses of the 2^l directional filters and their shifts:

$$\left\{ g_k^{(l)}[\cdot - S_k^{(l)} n] \right\}_{0\le k<2^l,\ n\in\mathbb{Z}^2}, \tag{2}$$

where the sampling matrices have the following two forms, depending on whether the representing direction is "nearly horizontal" or "nearly vertical":

$$\mathsf{S}_k^{(l)} = \begin{cases} \begin{bmatrix} 2^{l-1} & 0 \\ 0 & 2 \end{bmatrix} & 0 \le k < 2^{l-1}, \\ \begin{bmatrix} 2 & 0 \\ 0 & 2^{l-1} \end{bmatrix} & 2^{l-1} \le k < 2^l. \end{cases} \tag{3}$$

In the contourlet transform, suppose that an l_j-levels DFB is applied to the detail subspace W_j of the LP. This results in a decomposition of W_j into 2^{l_j} directional subspaces at scale 2^j:

$$W_j = \bigoplus_{k=0}^{2_j^l - 1} W_{j,k}^{(l_j)}. \tag{4}$$

Each subspace $W_{j,k}^{(l_j)}$ is spanned by a frame $\left\{\rho_{j,k,n}^{(l_j)}(t)\right\}_{n\in\mathbb{Z}^2}$ with a redundancy ratio equal to $4/3$, where

$$\rho_{j,k,n}^{(l_j)}(t) = \sum_{m\in\mathbb{Z}^2} g_k^{(l_j)}[m - \mathsf{S}_k^{(l_j)} n]\, \mu_{j,m}(t)$$

Furthermore, we can show that $\left\{\rho_{j,k,n}^{(l_j)}(t)\right\}_{n\in\mathbb{Z}^2}$ is generated from a single prototype function and its shifts as

$$\rho_{j,k,n}^{(l_j)}(t) = \rho_{j,k}^{(l_j)}(t - 2^{j-1}\mathsf{S}_k^{(l_j)} n), \quad \text{for all } n \in \mathbb{Z}^2 \qquad (5)$$

As a result, the $W_{j,k}^{(l_j)}$ is a shift invariant space which is defined on a rectangular grid of interval $2^{j+l_j-2} \times 2^j$ (or $2^j \times 2^{j+l_j-2}$, depending on the representing direction is nearly horizontal or nearly vertical). We refer to functions $\rho_{j,k,n}^{(l_j)}(t)$ as contourlets. The indexes j, k, and n are for the scale, direction, and location, respectively. Fig. 3 illustrates the subspaces and embedded grids of the contourlet expansion.

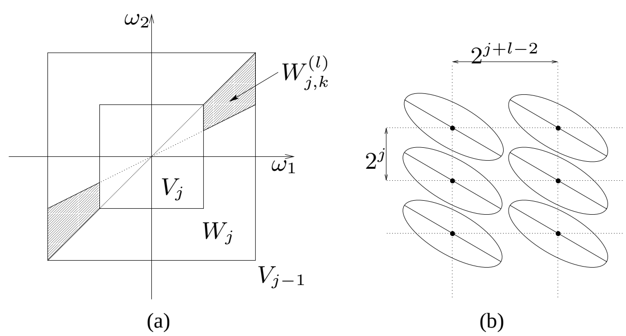

Fig. 3. (a) Multiscale and multidirectional subspaces generated by the contourlet construction. (b) Embedded grid of the subspace $W_{j,k}^{l}$.

In general, the contourlet construction allows for any number of DFB decomposition levels l_j to be applied at each LP level j. For the contourlet transform to satisfy the anisotropy scaling relation as in the curvelet transform, we simply need to impose that in the PDFB, the number of directions is doubled at every *other* finer scale of the pyramid. Fig. 4 graphically depicts the supports of the basis functions generated by such a PDFB. As can be seen from the two shown pyramidal levels, the support size of the LP is reduced by four times while the number of directions of the DFB is doubled. Combine these two steps, the support size of the PDFB basis functions are changed from one level to next in accordance with the curve scaling relation. In this contourlet scheme, each generation doubles the spatial resolution as well as the angular resolution.

4. NUMERICAL EXPERIMENTS

We now present several non-linear approximation experiments with the contourlet transform and compare it with the performance of a 2-D separable wavelet transform. In these NLA experiments, for a given value M, we select the M-most significant coefficients in each transform domain, and then evaluate the reconstructed images from these sets of M coefficients. The wavelet transform

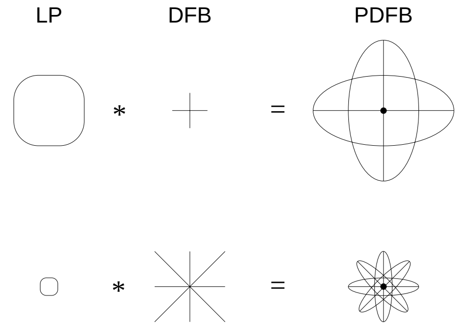

Fig. 4. Illustration of the supports for the contourlets implemented by a PDFB that satisfies the anisotropy scaling relation. From the upper line to the lower line, the scale is reduced by four while the number of directions is doubled.

used in the experiments is a biorthogonal transform with the "9-7" filters and 6 decomposition levels. The contourlet transform also uses the "9-7" filters in the LP stage, while the DFB stage uses the "23-45" biorthogonal quincunx filters designed by Phoong et al. [9]. The number of decomposition levels by the DFB at the finest pyramidal scale is 5, which leads to 32 directions.

Note that with this setup, both the wavelet and the contourlet transforms share the same multiscale detailed subspaces W_j, which are generated by the "9-7" filters. The difference is that in the wavelet transform, each subspace W_j is represented by a basis with three directions, whereas in the contourlet transform it is represented by a frame with many more directions. Since the two transforms share the same detailed subspaces, it is possible to restrict the comparison in these subspaces. We expect most of the refinement actions would happen around the image edges.

Fig. 5 and Fig. 6 show the sequences of non-linear approximated images at the finest subspace W_j using the wavelet and the contourlet transforms, respectively, where the input is the "Peppers" image. We observe that the wavelet scheme slowly refines the detailed image by isolated "dots" along the contours, while the contourlet scheme quickly refines by well-adapted "sketches". The improvement by the contourlet scheme can be seen both in terms of visual quality and the reconstruction error.

Finally, Fig. 7 shows a detailed comparison of two non-linear approximated images by the wavelet and contourlet transforms. We see that contourlets are superior compared with wavelets in capturing fine contours (directional textures on cloths).

5. CONCLUSION

In this work we constructed a discrete transform that can offers a sparse representation for piecewise smooth images. We first identified two extra features that could lead to an improvement over the wavelet scheme, namely directionality and anisotropy. From this, we proposed a new filter bank structure, the pyramidal directional filter bank (PDFB), that can provide a multiscale and directional decomposition for images with a small redundancy factor. The PDFB provides a frame expansion for images with frame elements like contour segments, and thus is also called the *contourlet transform*. The contourlet frame has small redundancy that is at most $4/3$. The connection between the developed discrete and continuous-domain constructions was made precisely via a new directional multiresolution analysis, which provides succes-

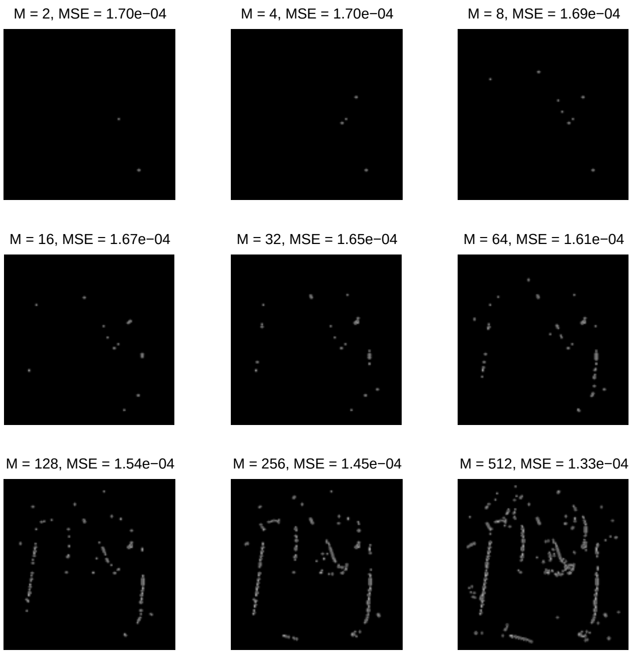

Fig. 5. Sequence of images showing the non-linear approximation at the finest scale of the wavelet transform. M is the number of the most significant coefficients; MSE is the mean square error against the projection of the input image into the finest detailed subspace. The input is the "Peppers" image.

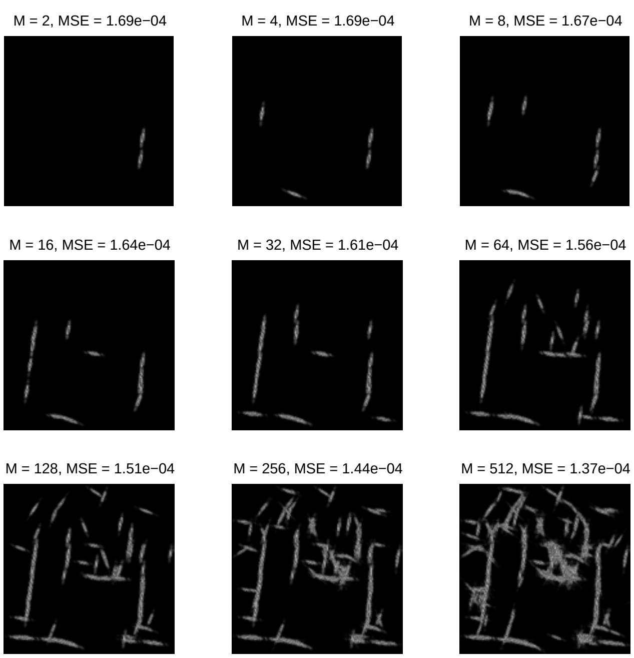

Fig. 6. Same as in Fig. 5 but with the contourlet transform. Note that the contourlet transform shares the same detailed subspace with the wavelet transform.

sive refinements at both spatial and directional resolutions. The

(a) Wavelet: PSNR = 24.34 dB (b) Contourlet: PSNR = 25.70 dB

Fig. 7. Detail of non-linear approximated images by the wavelet and contourlet transforms. In each case, the image originally of size 512×512 is reconstructed from the 4096-most significant coefficients in the transform domain.

contourlet transform can be designed to satisfy the anisotropy scaling relation for curves like the curvelet transform. Experiments with real images indicate the potential of the contourlets in image processing applications.

6. REFERENCES

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