ROBUST MULTICHANNEL SAMPLING

Ha T. Nguyen,

Sony Electronics Inc.,
Media Processing Technology Lab,
1730 N. First St., San Jose, CA 95112.
Email: thai-ha.nguyen@m4x.org.

Minh N. Do

University of Illinois at Urbana-Champaign,
Dept. of Electrical and Computer Engineering,
1406 W. Green St., Urbana, IL 61801.
Email: minhdo@uiuc.edu.

ABSTRACT

A conventional wisdom is that a *bandlimited* signal can be sampled at twice its maximum frequency to prevent any loss of information. For signals having high frequency components, sampling them requires fast analog-to-digital converters (ADC) that are difficult to design without increasing their cost and noise. In this paper, we show that high-resolution samples of any signal, bandlimited or unbandlimited, can be accurately approximated using multiple sequences of low-resolution samples taken from the same analog signal, probably with fractional delays, using slow ADCs. The approximation is enabled by designing a set of synthesis filters, without any knowledge of the signals to be sampled, to minimize an induced error system in the minimax sense. The approximation performance is guaranteed to be robust even when using estimates of the system parameters (such as antialiasing filters and fractional delays). We present experiments to confirm the potential of our approach.

Index Terms— Multichannel sampling, hybrid filter bank, sampled-data control, model-matching, filter design, $\mathcal{H}_\infty$ optimization.

1. INTRODUCTION

A conventional wisdom is that *bandlimited* signals can be sampled at twice its maximum frequency to prevent any loss of information [11]. There are two major drawbacks with this techniques. Firstly, the bandlimited assumption excludes a wide class of signals such as images. Secondly, designing analog-to-digital converters (ADC) for signals of large bandwidth is extremely challenging, often resulting in expensive and inaccurate ADCs [6, 14].

This paper addresses the above problems by proposing a multichannel sampling approach that allows to approximate a high-resolution digital signal, as if sampled from an analog signal by a fast ADC, using multiple low-resolution digital signals sampled by slow ADCs [7]. Figure 1(a) shows the model of a fast ADC used to obtain a *desired* high-resolution signal $y_0[n] = (f * \phi_0)(nh)$, for $n \in \mathbb{Z}$. An analog input signal $f(t)$ is convolved with an antialiasing filter, or sampling kernel function, $\phi_0(t)$ whose Laplace transform is $\Phi_0(s)$. The output of the convolution is then sampled at small sampling interval h, denoted by the operator S_h, i.e. $S_h\{v(t)\}[n] = v(nh)$.

In Figure 1(b), we depict how *actual* low-resolution signals $\{x_i[n]\}_{i=1}^N$ are sampled using slow ADCs. The same analog input $f(t)$ is sampled in parallel using N slow ADCs with antialiasing functions $\{\phi_i(t)\}_{i=1}^N$ (with Laplace transform $\{\Phi_i(s)\}_{i=1}^N$) and channel delays $\{D_i\}_{i=1}^N$. The low-resolution signals $x_i[n] = (f * \phi_i)(nMh - D_i)$, for $n \in \mathbb{Z}$, can be used to synthesize the high-resolution signal $y_0[n]$ of Fig. 1(a).

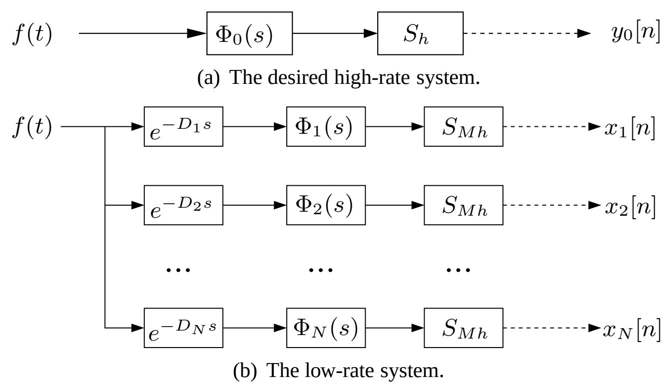

Fig. 1. (a) The desired high-rate system, (b) the low-rate system. The fast-sampled signal $y_0[n]$ can be approximated using slow-sampled signals $\{x_i[n]\}_{i=1}^N$.

The goal of the paper is to design synthesis filters $\{F_i(z)\}_{i=1}^N$ to approximate the high-resolution signal $y_0[n]$ using low-resolution signals $\{x_i[n]\}_{i=1}^N$, as shown in Fig. 2. The synthesis filters $\{F_i(z)\}_{i=1}^N$ are designed so that the *hybrid* induced error system $\mathcal{K}$ (Fig. 2) has the smallest error in the *minimax* sense. We note that, for clarity of presentation, we choose the implementation as in Fig. 2, although the polyphase technique [12, 13] can be used to offer parallelism and execution in the same clock speed of slow ADCs [7].

Among components of $\mathcal{K}$, the transfer functions $\{\Phi_i(s)\}_{i=0}^N$ characterize antialiasing filters, delays $\{D_i\}_{i=1}^N$ model system setup such as arrival times or sampling positions, and m_0 denotes the system delay for the high-rate signal $y_0[n]$ being approximated. We assume that functions $\{\Phi_i(s)\}_{i=0}^N$ and delays $\{D_i\}_{i=1}^N$ are measurable up to some errors. Multichannel sampling extends time-interleaved ADCs by allowing different antialiasing filters at slow ADCs [2]. Moreover, in many cases, the time delays $\{D_i\}_{i=1}^N$, although they can be measured [1], cannot be controlled. Under these conditions, the multichannel sampling setup studied in this paper can be ideally applied.

We note that many practical systems can be modeled as having rational transfer functions [5]. In the contrary, fractional delay operators e^{-Ds} are never rational if $D \neq 0$, though when D is an integer multiple of h, operator e^{-Ds} can be pushed after S_h to become an integer delay (in the digital domain). Working with fractional delay

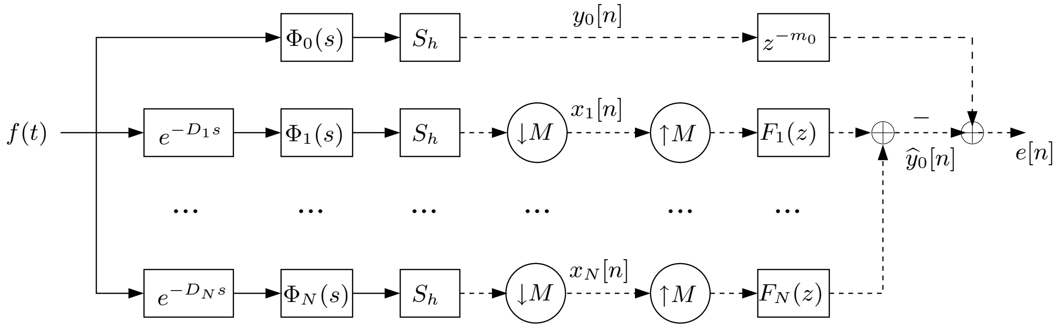

Fig. 2. The hybrid induced error system $\mathcal{K}$ with analog input $f(t)$ and digital output $e[n]$. We want to design synthesis filters $\{F_i(z)\}_{i=1}^N$ based on the transfer function $\{\Phi_i(s)\}_{i=0}^N$, the fractional delays $\{D_i\}_{i=1}^N$, the system delay tolerance m_0, the sampling interval h, and the super-resolution rate M to minimize the $\mathcal{H}_\infty$ norm of the induced error system $\mathcal{K}$.

operators $\{e^{-D_i s}\}$ is necessary, though nontrivial, to keep intersample behaviors of the input signals.

In Section 3, we prove that $\mathcal{K}$ is $\mathcal{H}_\infty$ norm equivalent to a finite-dimensional linear time-invariant (LTI) digital system. The conversion enables the design synthesis filters, IIR or FIR, to minimize the $\mathcal{H}_\infty$ norm of $\mathcal{K}$, using existing techniques such as model-matching and linear matrix inequalities. We show, in Section 4, the robustness of the designed induced error system $\mathcal{K}$ against estimate errors of the analysis filters $\{\Phi_i(s)\}_{i=1}^N$ and the delays $\{D_i\}_{i=1}^N$. More details on the presented techniques can be found in our previous publications [7, 8]. Experimental results are given in Section 5. Finally, we give conclusion in Section 6.

2. PROBLEM DEFINITION

Problem formulation. We consider the hybrid system $\mathcal{K}$ illustrated in Fig. 2. The $\mathcal{H}_\infty$ norm of $\mathcal{K}$ is defined as

$$\|\mathcal{K}\|_\infty := \sup_{f \in L^2, f \neq 0} \frac{\|e\|_2}{\|f\|_2}, \tag{1}$$

where $\|e\|_2$ is the l_2 norm of $e[n]$ and $\|f\|_2$ is the L_2 norm of $f(t)$. We want to design (IIR or FIR) synthesis filters $\{F_i(z)\}_{i=1}^N$ to minimize $\|\mathcal{K}\|_\infty$. The inputs of our algorithms consist of the strictly proper transfer functions $\{\Phi_i(s)\}_{i=0}^N$, the positive fractional delays $\{D_i\}_{i=1}^N$, the system delay tolerance $m_0 \geq 0$, the sampling interval $h > 0$, and the upsampling-rate $M \geq 2$.

In the design of the synthesis filter $\{F_i(z)\}_{i=1}^N$, the system performance is evaluated using the $\mathcal{H}_\infty$ approach [3, 4, 9]. In the digital-domain, we work on the Hardy space $\mathcal{H}_\infty$ that consists of all complex-value transfer matrices $\boldsymbol{G}(z)$ which are analytic and bounded outside of the unit circle $|z| > 1$. Hence $\mathcal{H}_\infty$ is the space of transfer matrices that are stable in the bounded-input bounded-output sense. The $\mathcal{H}_\infty$ norm of $\boldsymbol{G}(z)$ is defined as the maximum gain of the corresponding system. If a system $\boldsymbol{G}$, analog or digital, has input $\boldsymbol{u}$ and output $\boldsymbol{y}$, the $\mathcal{H}_\infty$ norm of $\boldsymbol{G}$ is [3]

$$\|\boldsymbol{G}\|_\infty = \sup \left\{ \|\boldsymbol{y}\|_2 : \boldsymbol{y} = \boldsymbol{G}\boldsymbol{u}, \|\boldsymbol{u}\|_2 = 1 \right\}. \tag{2}$$

The use of $\mathcal{H}_\infty$ optimization framework offers powerful tools for signal processing problems. In our case, the induced error is uniformly small over all finite energy inputs $f(t) \in \mathcal{L}_2(\mathbb{R})$ (i.e., $\|f(t)\|_2 < \infty$). In particular, no assumptions on the bandlimitedness of $f(t)$ are necessary. We minimize the worst induced error over all finite energy inputs $f(t)$. This is important since many practical signals are not bandlimited [10]. Finally, since $\mathcal{H}_\infty$ optimization is performed in the Hardy space, the designed filters are guaranteed to be stable.

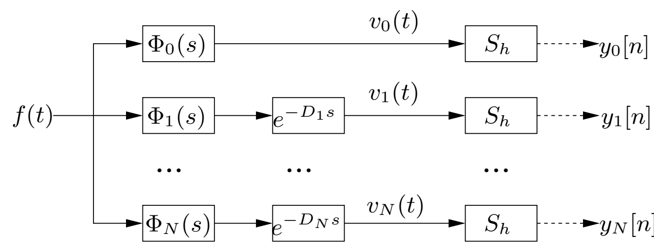

Fig. 3. The hybrid (analog input digital output) subsystem $\mathcal{G}$ of $\mathcal{K}$. Note that the sampling interval of all channels is h.

3. EQUIVALENCE TO A MODEL-MATCHING PROBLEM

Difficulties of designing the synthesis filters $\{F_i(z)\}_{i=1}^N$ include the hybrid nature of $\mathcal{K}$ and the fractional delays operators whose Laplace transforms $\{e^{-D_i s}\}_{i=1}^N$ are not rational functions if $D_i \neq 0$. In this section, we show that the hybrid system $\mathcal{K}$ of Fig. 2 is equivalent to an LTI digital system. The synthesis filters $\{F_i(z)\}_{i=1}^N$ then can be designed using traditional techniques such as model-matching and linear matrix inequalities (LMI) [7, 8].

We start by consider the analog part $\mathcal{G}$ of $\mathcal{K}$, as shown in Fig. 3. As the remaining part of $\mathcal{K}$ is digital, the key of the conversion to a digital system is to convert this analog part $\mathcal{G}$ using techniques in sampled-data control.

Proposition 1 *There exists a finite-dimensional digital system $\boldsymbol{G}_d$ that is $\mathcal{H}_\infty$ norm equivalent to $\mathcal{G}$.*

Note that, although the analog system $\mathcal{G}$ has a scalar input $f(t)$, the digital system $\boldsymbol{G}_d$ has multi-dimensional input $\boldsymbol{u}[n]$ of dimension n_u. Both $\mathcal{G}$ and $\boldsymbol{G}_d$ have $N+1$ output $\Big[y_0[n], y_1[n], \ldots, y_N[n]\Big]^T$. Replacing the system $\boldsymbol{G}_d$ into $\mathcal{K}$ of Fig. 2 we obtain an $\mathcal{H}_\infty$-norm equivalent system $\boldsymbol{K}_d$ as shown in Fig. 4. In the diagram of $\boldsymbol{K}_d$, subsystem $\boldsymbol{H}_i$, for $i = 0, 1, \ldots, N$, is the system with input $\boldsymbol{u}[n]$ and output $y_i[n]$. In general, $\{\boldsymbol{H}_i(z)\}_{i=1}^N$ are transfer matrices whose entries are IIR filters. Given the system $\boldsymbol{K}_d$ in Fig. 4, we can easily convert it into an LTI digital system, as in Fig. 5, using standard polyphase techniques [12, 13].

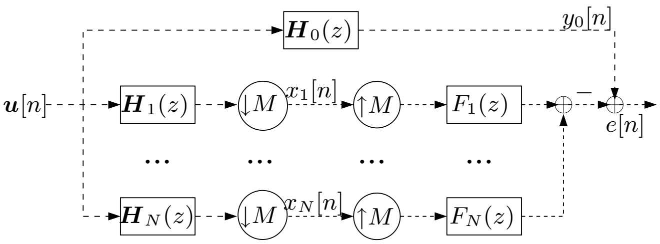

Fig. 4. The $\mathcal{H}_\infty$ norm equivalent digital system $\boldsymbol{K}_d$ of $\mathcal{K}$. Here $\{\boldsymbol{H}_i(z)\}_{i=0}^N$ are IIR transfer matrices. Note that the input $\boldsymbol{u}[n]$ is a multidimensional vector.

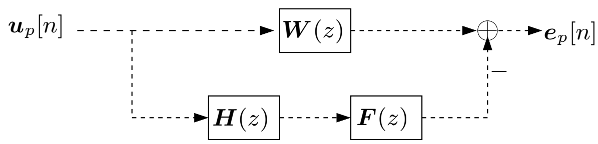

Fig. 5. The equivalent LTI error system $\boldsymbol{K}(z)$. Note that the system $\boldsymbol{K}(z)$ is Mn_u input M output, the transfer matrices $\boldsymbol{W}(z), \boldsymbol{H}(z)$ are of dimension $M \times Mn_u$, and $\boldsymbol{F}(z)$ is of dimension $M \times M$.

Theorem 1 *The digital error system $\boldsymbol{K}_d(z)$ is $\mathcal{H}_\infty$ norm equivalent to the LTI system*

$$\boldsymbol{K}(z) = \boldsymbol{W}(z) - \boldsymbol{F}(z)\boldsymbol{H}(z), \tag{3}$$

where $\boldsymbol{W}(z)$ and $\boldsymbol{H}(z)$ are transfer matrices that are determined from $\{\boldsymbol{H}_i(z)\}, M$, and N, and $\boldsymbol{F}(z)$, the polyphase representation of $\{F_i(z)\}_{i=1}^N$, is to be designed.

Using the result of Theorem 1, designing the synthesis filters $\{F_i(z)\}$ via $\boldsymbol{F}(z)$ to minimize $\|\boldsymbol{K}(z)\|_\infty$ is a traditional problem in control theory. Existing tools such as model-matching and linear matrix inequalities can be used to design IIR and FIR synthesis filters [3, 7, 8].

4. ROBUSTNESS

The conversion shown in Section 3 requires the knowledge of analysis filters $\{\Phi_i(s)\}_{i=0}^N$ and fractional delays $\{D_i\}_{i=1}^N$. In many practical applications, $\{\Phi_i(s)\}_{i=0}^N$ and $\{D_i\}_{i=1}^N$ can only be estimated. Suppose that $\{\widehat{\Phi}_i(s)\}_{i=1}^N$ and $\{\widehat{D}_i\}_{i=1}^N$ are used to design the synthesis filters $\{\widehat{F}_i(z)\}_{i=1}^N$. For simplicity, we assume that $\Phi_0(s)$ is perfectly known; for the case otherwise, the proof is similar to the techniques presented below. We show the robustness of the proposed design against estimate errors by proposing an upper bound of $\|\mathcal{K}\|_\infty$. We define operators

$$\begin{aligned} \boldsymbol{\Delta}(s) &= \text{diag}_N(e^{-(D_1-\widehat{D}_1)s}, \ldots, e^{-(D_N-\widehat{D}_N)s}), && (4)\\ \boldsymbol{\Phi}(s) &= \text{diag}_N(\Phi_1(s), \ldots, \Phi_N(s)) && (5)\\ \widehat{\boldsymbol{\Phi}}(s) &= \text{diag}_N(\widehat{\Phi}_1(s), \ldots, \widehat{\Phi}_N(s)), && (6) \end{aligned}$$

where $\text{diag}_N(\alpha_1, \ldots, \alpha_N)$ denotes a matrix with $\{\alpha_i\}$ in the diagonal, for operators $\{\alpha_i\}$, and zero elsewhere.

Let $\mathcal{W}$ represent the hybrid high-resolution channel of $\mathcal{K}$, and $\mathcal{F}$ signify the hybrid MIMO system composed of the delay operators $\{e^{-\widehat{D}_i s}\}_{i=1}^N$, the sampling operators S_{Mh}, upsampling by M, the synthesis filters $\{F_i(z)\}_{i=1}^N$, and the summation of N low-resolution channels. Then

$$\mathcal{K} = \mathcal{W} - \widehat{\mathcal{F}}\widehat{\boldsymbol{\Phi}}\boldsymbol{\Delta}. \tag{7}$$

Note that, in (7), the operator $\boldsymbol{\Delta}$ appears because operators $e^{-\widehat{D}_i s}$ are included in $\widehat{\mathcal{F}}$ instead of $e^{D_i s}$. It is easy to see that all the operators in (7), in particular $\widehat{\mathcal{F}}$, have bounded $\mathcal{H}_\infty$ norm. We assume that there exist positive constants δ_D and δ_Φ such that:

$$\begin{aligned} |D_i - \widehat{D}_i| &\leq \delta_D && (8)\\ \|\widehat{\boldsymbol{\Phi}}(s) - \boldsymbol{\Phi}(s)\|_\infty &\leq \delta_\Phi. && (9) \end{aligned}$$

Theorem 2 *The induced error system $\mathcal{K}$ is robust in the sense that its $\mathcal{H}_\infty$ norm is bounded as*

$$\|\mathcal{K}\|_\infty \leq \|\mathcal{W} - \widehat{\mathcal{F}}\widehat{\boldsymbol{\Phi}}\|_\infty + \delta_\Phi \cdot \|\widehat{\mathcal{F}}\|_\infty + \sqrt{\delta_D} \cdot C \cdot \|\mathcal{F}\|_\infty, \tag{10}$$

for some constant C independent of δ_D and δ_Φ.

Proof Indeed:

$$\begin{aligned} \|\mathcal{K}\|_\infty &= \|\mathcal{W} - \widehat{\mathcal{F}}\boldsymbol{\Phi}\boldsymbol{\Delta}\|_\infty \\ &\leq \|\mathcal{W} - \widehat{\mathcal{F}}\widehat{\boldsymbol{\Phi}}\|_\infty + \|\widehat{\mathcal{F}}\widehat{\boldsymbol{\Phi}} - \widehat{\mathcal{F}}\boldsymbol{\Phi}\|_\infty + \|\widehat{\mathcal{F}}\boldsymbol{\Phi} - \widehat{\mathcal{F}}\boldsymbol{\Phi}\boldsymbol{\Delta}\|_\infty \\ &\leq \|\mathcal{W} - \widehat{\mathcal{F}}\widehat{\boldsymbol{\Phi}}\|_\infty + \delta_\Phi \cdot \|\widehat{\mathcal{F}}\|_\infty + \sqrt{\delta_D} \cdot C \cdot \|\mathcal{F}\|_\infty. \end{aligned}$$

The last inequality is derived from (9) and our previous result on the robustness against delay jitters $(D_i - \widehat{D}_i)$ [7]. ■

From the result of Theorem 2, we see that using estimated analysis filters and fractional delays causes additional terms, i.e. the second and the third terms in (10). Specifically, the second term in (10) represents the error caused by estimate of the analysis filters, and the third term addresses the error of the fractional delay estimates.

5. EXPERIMENTS

In this section, we use $N = 8$ low-resolution signals to increase $M = 5$ times the resolution. We also use $m_0 = 10$ and $h = 1$. The fractional delays $\{D_i\}_{i=1}^N$ are randomly chosen in $[0, Mh]$. All the analysis antialiasing filters are chosen as

$$\Phi(s) = \frac{0.25}{s^2 + s + 0.25}. \tag{11}$$

As input, we use the following an unbandlimited signal:

$$f(t) = \begin{cases} 0 & t < 0.3, \\ 1 & t \geq 0.3. \end{cases} \tag{12}$$

We compare the proposed algorithm to the following method, called Sinc method. Each channel uses a filter derived from the function $\text{sinc}(x) = \sin(\pi x)/(\pi x)$ to approximately undo the effect of fractional delays as follows:

$$F_i^{(\text{sinc})}[n] = \text{sinc}\left(n - \frac{D_i}{Mh}\right). \tag{13}$$

The high-resolution signal is obtained by interleaving the best, chosen based on the delays, $M = 5$ channels after the filtering process.

In Fig. 6, we show the approximation errors of the high resolution signals for both methods. The maximum error is 0.0029, and the mean square error is 3.7534×10^{-7} for the proposed method. For Sinc method, the maximum is 0.1636 and the the mean square error is 1.9596×10^{-004}. The induced error is $\|\mathcal{K}\|_\infty = 0.03776$. The proposed method is observed to outperform the Sinc approach for it

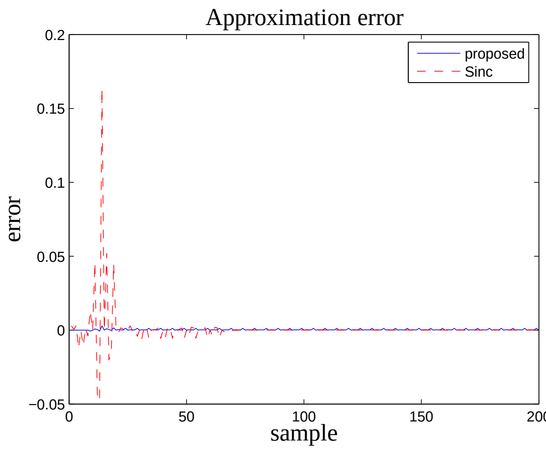

Fig. 6. The approximation error for a step function as input. We use $N = 8$ channels to synthesize a signals of $M = 5$ times higher resolution. The maximum error and mean square error are 0.0029 and 3.7534×10^{-7} for the proposed method. For Sinc method, these numbers are 0.1636 and 1.9596×10^{-4}, respectively.

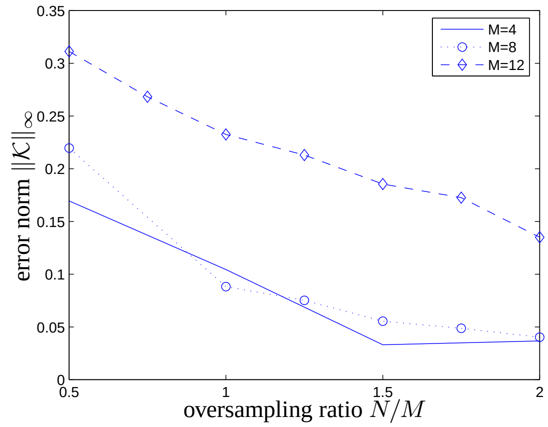

Fig. 7. The median value of $\|\mathcal{K}\|_\infty$ for ten runs plotted against the oversampling ratio N/M, for $M = 4, 8, 12$.

takes all the information into account to approximate the high resolution signal. In particular, the proposed method, without modifying the framework, can easily take into account additional information in case of oversampling $N > M$.

In Fig. 7 we plot $\|\mathcal{K}\|_\infty$ against the oversampling ratio N/M, for $M = 4, 8, 12$. For each pair of values M and N, ten simulations are run for $\{D_i\}$ randomly chosen in $[0, Mh]$. The median value of $\|\mathcal{K}\|_\infty$ for these ten runs is used in the plots of Fig. 7. We observe that the error does decrease when more low-resolution channels are used, although for higher values of M, to achieve the same error (in term of the induced error system's $\mathcal{H}_\infty$ norm) the oversampling ratio tends to be higher.

6. CONCLUSIONS

We presented in this paper a multichannel sampling technique that synthesizes high-resolution signals using multiple low-resolution signals sampled from the same analog input signal (with different fractional delays). Our approach does not put any assumption on the input analog signal such as bandlimitedness, but instead minimize the worst error gain in squared norm. We showed that the design of the synthesis filters is equivalent to a traditional model-matching problem and proved that the approximation error, in the minimax sense, is guaranteed to be robust even in the presence of estimate errors of the antialiasing filters and the fractional delays. Experiments showed the proposed approach outperformed a traditional approach using sinc function. For future works, we want to extend the approach to the case where both $\{\Phi_i(s)\}$ and $\{D_i\}$ are (slowly) time-variant.

7. REFERENCES

[1] J. Benesty, J. Chen, and Y. Huang, "Time-delay estimation via linear interpolation and cross correlation," *IEEE Trans. Speech Audio Proc.*, vol. 12, no. 5, pp. 509–519, September 2004.

[2] J. Franca, A. Petraglia, and S. K. Mitra, "Multirate analog-digital systems for signal processing and conversion," in *Proc. IEEE*, vol. 35, no. 2, February 1997, pp. 242–262.

[3] B. Francis, *A Course in $\mathcal{H}_\infty$ Control Theory*. Heidelberg, Germany: Springer-Verlag, 1987.

[4] M. Green and D. J. N. Limebeer, *Linear Robust Control*. Upper Saddle River, NJ: Prentice-Hall, Inc., 1995.

[5] B. P. Lathi, *Linear Systems and Signals*, 2nd ed. New York, NY: Oxford University Press, 1992.

[6] B. Le, T. W. Rondeau, J. H. Reed, and C. W. Bostian, "Analog-to-digital converters," *IEEE Signal Proc. Mag.*, vol. 22, no. 6, pp. 69–77, November 2005.

[7] H. T. Nguyen and M. N. Do, "Hybrid filter bank with fractional delays: Minimax design and application to multichannel sampling," *IEEE Trans. Signal Proc.*, to appear.

[8] ——, "Signal reconstruction from a periodic nonuniform set of samples using $\mathcal{H}_\infty$ optimization," in *Proc. of SPIE*, vol. 6498, San Jose, February 2007.

[9] H. Shu, T. Chen, and B. Francis, "Minimax design of hybrid multirate filter banks," *IEEE Trans. Circ. and Syst.*, vol. 44, no. 2, February 1997.

[10] D. Slepian, "On bandwidth," in *Proc. IEEE*, vol. 64, no. 3, 1976, pp. 292–300.

[11] M. Unser, "Sampling - 50 years after Shannon," *Proc. IEEE*, vol. 88, no. 4, pp. 569–587, April 2000.

[12] P. P. Vaidyanathan, *Multirate Systems and Filter Banks*. New York, NY: Prentice Hall, 1993.

[13] M. Vetterli and J. Kovačević, *Wavelets and Subband Coding*. New York, NY: Prentice-Hall, 1995.

[14] R. H. Walden, "Performance trends for analog to digital converters," *IEEE Communications Magazine*, vol. 37, no. 2, pp. 96–101, Febfruary 1999.